Chapter three

$continuity$:

$definition$: $let\ \delta\ be\ defined\ on\ a\ nbhd\ of\ z_0\ then\ f\ is\ said\ to\ be\ continuous\ at\ z_0$

$if\ \lim_{z\to z_0} f(z) = f(z_0)$

$notice\ that\ in\ it\ ,we\ have$

1. $\lim_{z\to z_0} f(z)\ exist$
2. $f(z_0)\ exist$
3. $\lim_{z\to z_0} f(z) = f(z_0)$

$so\ f\ is\ continnous\ at\ z_0\ if\ for\ each\ \epsilon > 0\ \exists\ \delta \equiv \delta\ (\epsilon, z_0) > 0$

$such\ that\ |z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \epsilon$

$if\ f\ is\ continuous\ at\ each\ point\ of\ a\ region\ \mathbb{C}\ then\ we\ have\ say\ that\ f\ is\ continuous$

$on\ \mathbb{C}$.

$example$:

1. $suppose\ \ f(z) = x^2 - y^2 + ixysin\overline{x}$

   $then\ f\ is\ continuous\ on\ \mathbb{C}$

2. $suppose\ f(z) = e^{xy} - e^{y^2/2} + i\sin(x^2 - 2y + 3)$

   $then\ f\ is\ continuous\ on\ \mathbb{C}$

$lemma : -$

$suppose\ f, g\ are\ contiuous\ function\ on\ R\ \ then\ f + g\ ,f$
$$- g\ , f.g\ , \frac{f}{g} (g \neq 0)$$

$are\ all\ continuous\ functions\ on\ R.$

$lemma : -$

$composite\ two\ continuous\ function\ is\ again\ a\ continuous\ function\ .$

$example$:

e^{2z} *is continuous on* $\mathbb{C}$.

$Note: -$

suppose f *is continuos on* C

$|f(z)|$, $f(\bar{z})$, $\overline{f(z)}$ *are continuos function on* C .

example

suppose $f(z) = 2z^2 + iz$ *then* f *is continuous on* $\mathbb{C}$,

because it is polynomial

example:

suppose $f(z) = \frac{z}{z^2+4}$, *it is continuous on* $\mathbb{C}$,*except* $z = \pm 2i$

example:

$f(z) = \frac{\bar{z}}{z}$, *it is not continuous when* $z = 0$

since $\lim_{z \to 0} \frac{\bar{z}}{z}$ *is not exist* .

$DERIVATIVIES: -$

let f *be a function defined on adomain containing anbhd of apoint* z_0 ,*then the derivative of* f *at* z_0 ,*denoted by* $\grave{f}(z_0)$ *is defined by*

$$\grave{f}(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

provided the limit exists .

f *is said to be differentiable at* z_0 *if the derivative* $\grave{f}(z_0)$ *exists* .

set , $\Delta z = z - z_0$

$$\text{then } \grave{f}(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

equavalently,

$$\grave{f}(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$here\ |\Delta z|\ is\ so\ small\ that\ f(z_0 + \Delta z) is\ defined.$

$example: -$

$f(z) = z^2 + z + i\ \ , for\ the\ derivative$

$f(z+\Delta z) - f(z) = (z+\Delta z)^2 + (z+\Delta z) + i - (z^2 + z + i)$

$= (z+\Delta z)^2 + z + \Delta z - (z^2 + z)$

$= z^2 + 2z\Delta z + (\Delta z)^2 + z + \Delta z - z^2 - z$

$= \Delta z(2z + \Delta z + 1)$

$so, \grave{f}(z) = \lim_{\Delta z \to 0} \frac{\Delta z(2z+\Delta z+1)}{\Delta z}$

$= \lim\limits_{\Delta z \to 0} (2z + \Delta z + 1)$

$= 2z + 1$

$example(2): -$

$consider\ f(z) = |z|^2\ , if\ z = x + iy\ , then\ f(z) = |z|^2 = x^2 + y^2$

$= x^2 + y^2 + 0.i$

$= u(x,y) + iv(x,y)$

$where\ u(x,y) = x^2 + y^2$

$v(x,y) = 0$

$since\ u\ and\ v\ are\ continuous\ on\ \mathbb{C}\ it\ follows\ that\ f\ is\ continuous\ on\ \mathbb{C}$

$for\ the\ differentability\ observe\ that$

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{|z+\Delta z|^2 - |z|^2}{\Delta z}$$

$$= \frac{(z+\Delta z)(\overline{z+\Delta z}) - z\bar{z}}{\Delta z} = \frac{(z+\Delta z)(\bar{z} + \Delta\bar{z}) - z\bar{z}}{\Delta z}$$

$$= \frac{z\bar{z} + z\Delta\bar{z} + \bar{z}\Delta z + \Delta z\Delta\bar{z} - z\bar{z}}{\Delta z}$$

$$= z\frac{\Delta\bar{z}}{\Delta z} + \bar{z} + \Delta\bar{z}$$

$let\ \Delta z\ be\ the\ corresponding\ increment\ in\ z\ for\ the\ increments\ (\Delta x, \Delta y), or$

$\Delta z = \Delta x + i\Delta y$

$for\ the\ limit\ we\ choose\ the\ paths$

1. $(\Delta x, \Delta y) \to (\Delta x, 0) \to (0,0)$
2. $(\Delta x, \Delta y) \to (0, \Delta y) \to (0,0)$

$Along\ the\ part\ (1) as\ \Delta z \to 0$

$we\ get$

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} = \bar{z} + \frac{z(\Delta x - i\Delta y)}{(\Delta x + i\Delta y)} + (\Delta x - i\Delta y)$$

$= \bar{z} + z + \Delta x$

$as\ (\Delta x, \Delta y) \to (\Delta x, 0)$

$= \bar{z} + z$

$as\ (\Delta x, \Delta y) \to (\Delta x, 0) \to (0,0)$

$simillarly\ \ , alongs\ the\ path\ (2).$

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} = \bar{z} + \frac{z(\Delta x - i\Delta y)}{(\Delta x + i\Delta y)} + \Delta x - i\Delta y$$

$= \bar{z} + z(-1) - i\Delta y$

$as\ (\Delta x, \Delta y) \to (0, \Delta y)$

$= \bar{z} - z$

$as\ (\Delta x, \Delta y) \to (0, \Delta y) \to (0,0)$

$for\ the\ uniqueness\ of\ the\ limit$

$\bar{z} + z = \bar{z} - z \Rightarrow z = 0$

so

$f(\ z) = |z|^2$ $is\ diffrentiable\ only\ at\ z = 0\ where\ its\ dervative$

$\acute{f}\ (z) = 0\ \ as\ z = 0\ this\ function\ is\ not\ differentiable\ at\ any$

$point\ z \neq 0$

$lemma: -$

$a\ differentiable\ function\ is\ always\ continuos\ .$

$*\ differentiation\ formulas: -$

$the\ definition\ of\ derivative$

$$\acute{f}(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$is\ identical\ in\ form\ to\ that\ of\ the\ derivative\ of\ areal\ value\ function\ of\ area$

$variable\ .$

$in\ fact\ ,the\ basic\ differentiation\ formulas\ given\ below\ can\ be\ derived\ from$

$that\ definition\ together\ with\ various\ theorems\ on\ limits\ ,by\ essentially$

$the\ same\ steps\ as\ the\ ones\ used\ in\ calculus.in\ these\ formulas.$

$the\ derivative\ of\ a\ function\ f\ at\ a\ point\ z\ is\ denoted\ by\ either$

$\acute{f}(z)\ \ or\ \ \frac{d}{dz} f(z)$

$let\ c\ be\ a\ complex\ constant\ ,and\ let\ f\ be\ a\ funnction\ whose\ derivative$

$exists\ at\ a\ point\ z\ ,it\ is\ easy\ to\ show\ that\ .$

$lemma: -$

$1.\frac{d}{dz}(c) = 0\ ,\frac{d}{dz}(z) = 1$

$\frac{d}{dz}\big(c\ f(z)\big) = c\ \acute{f}(z)$

$$2.\frac{d}{dz}(z^n) = n\,z^{n-1}$$

$$3.\frac{d}{dz}[f(z) \pm g(z)] = \acute{f}(z) \pm \acute{g}(z)$$

$$4.\frac{d}{dz}\big(f(z)g(z)\big) = f(z)\acute{g}(z) + g(z)\acute{f}(z)$$

$$5.\frac{d}{dz}\left(\frac{f(z)}{g(z)}\right) = \frac{g(z)\acute{f}(z) - f(z)\acute{g}(z)}{(g(z))^2}$$