2-Nonhomogeneous linear equations with constant coefficients

A second order nonhomogeneous equation with constant coefficients is written as

$$ay'' + by' + cy = R(x) \quad \cdots (1)$$

where $a,\ b$ and c are constant $a \neq 0$, $R(x) \neq 0$.

I- Undetermined Coefficients

We use the method of undetermined coefficients for finding particular solutions y_p when $R(x)$ is one of the forms shown in table.

$R(x)$	Notes	Form of y_p
ax^n	$n \geq 0$	$A_n x^n + A_{n-1}x^{n-1} + \cdots + A_2x^2 + A_1x + A_0$
ae^{rx}	$r \neq m_1 \neq m_2$	Ae^{rx}
	$r = m_1$ or $r = m_2$	Axe^{rx}
	$r = m_1 = m_2$	Ax^2e^{rx}
$a\sin kx$ or	$k \neq \beta$	$A\sin kx + B\cos kx$
	$k = \beta$	$x(A\sin kx + B\cos kx)$

We write the general solution of (1) as the sum of the homogeneous y_h and particular solutions y_p. $\quad y = y_h + y_p$

Example 1: Solve $\quad y'' - y' - 12y = 36x - 15$

Solution: First, we solve the homogeneous equation. The characteristic equation is

$$m^2 - m - 12 = 0 \Rightarrow (m+3)(m-4) = 0 \quad \Rightarrow \quad m_1 = -3 \quad , \quad m_2 = 4 \square$$

$$y_h = c_1e^{-3x} + c_2e^{4x} \square$$

Second, let $y_p = Ax + B$ then $y_p' = A$ and $y_p'' = 0$ □

Upon substitution into the differential equation, we have

$$0 - A - 12(Ax + B) = 36x - 15 \quad \Rightarrow \quad \underbrace{-12Ax}\,\underbrace{-A - 12B} = \underbrace{36x}\,\underbrace{-15} \quad \square$$

$$-12A = 36 \Rightarrow A = -3 \quad \text{and} \quad -A - 12B = -15 \quad \Rightarrow \quad B = 1$$

So $\quad y_p = -3x + 1$

Then $\quad y = y_h + y_p = c_1e^{-3x} + c_2e^{4x} - 3x + 1 \square$

Example 2: Solve $y'' - 2y' + 5y = 10x^2 + 7x$

Solution: First, we solve the homogeneous equation. The characteristic equation is

$$m^2 - 2m + 5 = 0 \quad \square$$

$$m = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a} = \frac{-(-2) \mp \sqrt{(2)^2 - 4 \times 1 \times 5}}{2 \times 1} = 1 \mp 2i$$

$$y_h = e^x(c_1 \sin 2x + c_2 \cos 2x)$$

Second, let $y_p = Ax^2 + Bx + C$ then $y_p' = 2Ax + B$ and $y_p'' = 2A$ □

Upon substitution into the differential equation, we have

$$2A - 4Ax - 2B + 5Ax^2 + 5Bx + 5C = 10x^2 + 7x \square$$

$$\underline{\underline{5Ax^2}} + \underline{(-4A + 5B)x} + (2A - 2B + 5C) = \underline{\underline{10x^2}} + \underline{7x} + 0 \square$$

Equating coefficient x^2on the left side with the coefficient x^2 in the right side we get

$$5A = 10 \quad \Rightarrow \quad A = 2$$

By the same way equate coefficient x : $-4A + 5B = 7 \Rightarrow B = 3$

And $2A - 2B + 5C = 0 \Rightarrow C = 2/5$

So $y_p = 2x^2 - 3x + 2/5$

$$\therefore \quad y = y_h + y_p = e^x(c_1 \sin 2x + c_2 \cos 2x) + 2x^2 + 3x + 2/5 \square$$

Example 3: Solve $y'' - y' - 6y = 8e^{2x}$

Solution: First, we solve the homogeneous equation. The characteristic equation is

$$m^2 - m - 6 = 0 \Rightarrow (m-3)(m+2) = 0 \Rightarrow m_1 = 3 \ , \quad m_2 = -2 \square$$

$$y_h = c_1 e^{3x} + c_2 e^{-2x}$$

Second, we find a particular solution of the nonhomogeneous equation. The form of the particular solution is chosen such that the exponential will cancel out of both sides of the differential equation. We choose

$$y_p = Ae^{2x} \Rightarrow y_p' = 2Ae^{2x} \Rightarrow y_p'' = 4Ae^{2x}$$

$$4A - 2A - 6A = 8 \Rightarrow A = -2$$

So $y_p = -2e^{2x}$

Then $y = y_h + y_p = c_1 e^{3x} + c_2 e^{-2x} - 2e^{2x}$

Example 4: Solve $y'' - 5y' + 6y = 2e^{3x}$

Solution: $m^2 - 5m + 6 = 0 \Rightarrow (m-3)(m-2) = 0 \quad \Rightarrow m_1 = 3 \;,\; m_2 = 2$

$$y_h = c_1 e^{3x} + c_2 e^{2x}$$

$m_1 = r = 3$ so $y_p = Axe^{3x}$

$$y_p' = 3Axe^{3x} + Ae^{3x}$$

$$y_p'' = 9Axe^{3x} + 3Ae^{3x} + 3Ae^{3x} = 9Axe^{3x} + 6Ae^{3x}$$

$$9Axe^{3x} + 6Ae^{3x} - 15Axe^{3x} - 5Ae^{3x} + 6Axe^{3x} = 2e^{3x}$$

$$A = 2$$

So $y_p = 2e^{3x}$

Then $y = y_h + y_p = c_1 e^{3x} + c_2 e^{2x} + 2e^{3x}$

Example 5: Solve $y'' - 3y' - 4y = 2\sin 2x$

Solution: The characteristic equation is $m^2 - 3m - 4 = 0 \Rightarrow m_1 = 4 \;,\; m_2 = -1$

$$y_h = c_1 e^{4x} + c_2 e^{-x}$$

Now let $y_p = A\sin 2x + B\cos 2x$

$y_p' = 2A\cos 2x - 2B\sin 2x$ and $y_p'' = -4A\sin 2x - 4B\cos 2x$

Upon substitution into the differential equation, we obtain

$$\underline{-4A\sin 2x}\ \underbrace{-4B\cos 2x - 6A\cos 2x} + \underline{6B\sin 2x - 4A\sin 2x}\ \underbrace{-4B\cos 2x} = \underline{2\sin 2x}$$

$$-8A + 6B = 2 \quad \Rightarrow \quad -4A + 3B = 1$$

$$-6A - 8B = 0 \quad \Rightarrow \quad -3A - 4B = 0$$

$$A = \frac{\begin{vmatrix} 1 & 3 \\ 0 & -4 \end{vmatrix}}{\begin{vmatrix} -4 & 3 \\ -3 & -4 \end{vmatrix}} = -\frac{4}{25} \quad \text{and} \quad B = \frac{\begin{vmatrix} -4 & 1 \\ -3 & 0 \end{vmatrix}}{\begin{vmatrix} -4 & 3 \\ -3 & -4 \end{vmatrix}} = \frac{3}{25}$$

$$y_p = -\frac{4}{25}\sin 2x + \frac{3}{25}\cos 2x$$

So that $y = y_h + y_p = c_1 e^{4x} + c_2 e^{-x} - \frac{4}{25}\sin 2x + \frac{3}{25}\cos 2x$

Example 6: Solve $y'' - 8y' + 16y = 23\cos x - 7\sin x$

Solution: The characteristic equation is $m^2 - 8m + 16 = 0 \Rightarrow m_1 = m_2 = 4$

$$y_h = (c_1 x + c_2)e^{4x}$$

Now let $y_p = A\sin x + B\cos x$

$$y_p' = A\cos x - B\sin x \quad \text{and} \quad y_p'' = -A\sin x - B\cos x$$

$$-A\sin x - B\cos x - 8A\cos x + 8B\sin x + 16A\sin x + 16B\cos x$$

$$= 23\cos x - 7\sin x$$

$$15A + 8B = -7 \quad \text{and} \quad -8A + 15B = 23$$

$$A = \frac{\begin{vmatrix} -7 & 8 \\ 23 & 15 \end{vmatrix}}{\begin{vmatrix} 15 & 8 \\ -8 & 15 \end{vmatrix}} = \frac{-289}{289} = -1 \quad \text{and} \quad B = \frac{\begin{vmatrix} 15 & -7 \\ -8 & 23 \end{vmatrix}}{\begin{vmatrix} 15 & 8 \\ -8 & 15 \end{vmatrix}} = \frac{289}{289} = 1$$

$$y_p = -\sin x + \cos x$$

$$y = y_h + y_p = (c_1 x + c_2)e^{4x} - \sin x + \cos x$$

Example 7: Solve $y'' - 6y' + 13y = 2\sin x - 3\cos x$

Solution: $m^2 - 6m + 13 = 0$

$$m = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a} = \frac{-(-6) \mp \sqrt{(-6)^2 - 4 \times 1 \times 13}}{2 \times 1} = 3 \mp 4i$$

$$y_h = e^{3x}(c_1 \sin 4x + c_2 \cos 4x)$$

Now let $y_p = A\sin x + B\cos x$

$$y_p' = A\cos x - B\sin x \quad \text{and} \quad y_p'' = -A\sin x - B\cos x$$

$$-A\sin x - B\cos x - 6A\cos x + 6B\sin x + 13A\sin x + 13B\cos x = 2\sin x - 3\cos x$$

$$12A + 6B = 2 \Rightarrow 6A + 3B = 1$$

$$-6A + 12B = -3 \Rightarrow -2A + 4B = -1$$

$$A = \frac{\begin{vmatrix} 1 & 3 \\ -1 & 4 \end{vmatrix}}{\begin{vmatrix} 6 & 3 \\ -2 & 4 \end{vmatrix}} = \frac{7}{30} \quad \text{and} \quad B = \frac{\begin{vmatrix} 6 & 1 \\ -2 & -1 \end{vmatrix}}{\begin{vmatrix} 6 & 3 \\ -2 & 4 \end{vmatrix}} = \frac{-4}{30} = -\frac{2}{15}$$

$$y_p = (7/30)\sin x - (2/15)\cos x$$

$$y = y_h + y_p = e^{3x}(c_1 \sin 4x + c_2 \cos 4x) + (7/30)\sin x - (2/15)\cos x$$

Example 8: Solve $y'' + y = 4\sin x$

Solution: $m^2 + 1 = 0 \quad \rightarrow \quad m = \pm i$

$$y_h = c_1 \sin x + c_2 \cos x$$

$$k = \beta = 1$$

$y_p = x(A\sin x + B\cos x)$

$y_p' = x(A\cos x - B\sin x) + A\sin x + B\cos x$

$y_p'' = x(-A\sin x - B\cos x) + A\cos x - B\sin x + A\cos x - B\sin x$

$= -x(A\sin x + B\cos x) + 2A\cos x - 2B\sin x$

Upon substitution into the differential equation, we obtain

$-x(A\sin x + B\cos x) + 2A\cos x - 2B\sin x + x(A\sin x + B\cos x) = 4\sin x$

$2A\cos x - 2B\sin x = 4\sin x \quad \Rightarrow \quad A = 0 \quad and \quad -2B = 4 \quad \Rightarrow \quad B = -2$

$y_p = -2x\cos x$

$y = y_h + y_p = c_1\sin x + c_2\cos x - 2x\cos x$

Exercises

For each of the following problem solve the differential equation

(1) $y'' - 2y' + y = x^2 - x - 3$

(2) $y'' + 7y' + 12y = 4e^{2x}$ □

(3) $y'' + 8y' + 12y = 3\cos x$ □

(4) $y'' - 2y' + 10y = 54e^{-2x}$ □

(5) $y'' + 2y' + 2y = \sin 2x + 5\cos 2x$ □

(6) $y'' - 6y' + 9y = 4\sin x - 3\cos x$□

(7) $y'' + 4y' + 3y = 4e^{-x}$□

(8) $y'' + 4y = 2\cos 2x$□