**Method of halving the interval (Bisection Method):**

**Suppose a continuous function f defined on the interval $[x_0, x_1]$ is given with $f(x_0)$ and $f(x_1)$ of opposite sign (i.e. $f(x_0)\times f(x_1)<0$).** Then by intermediate value theorem ((If $f\in C[x_0, x_1]$ and k is any number between $f(x_0)$ and $f(x_1)$, then there exists $x_2\in(x_0, x_1)$ for which $f(x_2)=k$)) there exists a point $x_2\in(x_0, x_1)$ such that $f(x_2)=0$ **(there is at least root of $f(x)=0$ ). We now bisect this interval $x_2 = \frac{x_0+x_1}{2}$** , and then three possibilities arises( **and then we consider the interval [ $x_0$ , $x_2$ ] or [ $x_2$ , $x_1$ ] at whose end- point take apposite sign** )

$$\text{If } f(x_0)\times f(x_2) = \begin{cases} <0 & \text{there is a root between } x_0 , x_2 \Rightarrow x_3=\frac{x_0+x_2}{2} \\ >0 & \text{there is a root between } x_1 , x_2 \Rightarrow x_3=\frac{x_1+x_2}{2} \\ =0 & x_2 \text{ is exact root ((Stop)).} \end{cases}$$

**We stop iteration if the interval width is as small as desired i.e. $|x_i - x_{i-1}| \le \varepsilon$ for any i.**

**Example1 :-**

Use six iteration to locate the root of equation $2^x - 5x + 2$ by using bisection method

Solution :

| | | |
|---|---|---|
| $f(0) = 3$ | | $f(1) = -1$ |
| $f(0) = 3$ | $f(0.5) = 0.914$ | $f(1) = -1$ |
| $f(0.5) = 0.914$ | $f(0.75) = -0.068$ | $f(1) = -1$ |
| $f(0.5) = 0.914$ | $f(0.625) = 0.417$ | $f(0.75) = -0.068$ |
| $f(0.625) = 0.417$ | $f(0.6875) = 0.1729$ | $f(0.75) = -0.068$ |
| $f(0.6875) = 0.1729$ | $f(0.71875) = 0.052$ | $f(0.75) = -0.068$ |
| $f(0.71875) = 0.052$ | $f(0.734375) = -0.008$ | $f(0.75) = -0.068$ |

$\therefore$ The root is 0.734375

**Example 2:**

Find an approximate root of $f(x)=x^2-2$ in the interval [1,2] by using Bisection method if its possible with error $\varepsilon \le 10^{-4}$

**Solution:** It is possible to use bisection method because f is continuous on [1,2] and f(a)=f(1)=-1; f(b)=f(2)=2 i.e. f(a)×f(b)=-2<0.

$x_1=\frac{a+b}{2}=\frac{1+2}{2}=1.5$, $|a-x_1|=0.5>\varepsilon$. Find $x_2$:

$f(x_1)=0.25 \Rightarrow f(x_1)\times f(a)<0 \Rightarrow$ there is a root between $x_1$ and a $\Rightarrow x_2=\frac{x_1+a}{2}=1.25$

$|x_2-x_1|=0.25>\varepsilon$. Find $x_3$:

$f(x_2)=-0.437 \Rightarrow f(x_2)\times f(x_1)<0 \Rightarrow$ there is a root between $x_1$ and $x_2 \Rightarrow x_3=\frac{x_1+x_2}{2}=1.375$

$\vdots$

Stop iteration if $|\boldsymbol{x_i}-\boldsymbol{x_{i-1}}|\le\boldsymbol{\varepsilon}$ for any i=1, 2, ….

**Example 3:**

Find an approximate root of f(x)=xlog(x)-1=0 in the interval [1,2] by using Bisection method with error $\varepsilon \le 10^{-3}$

**Solution:** $f(x_0)=f(1)=-1$; $f(x_1)=f(2)=0.3863$ i.e. $f(x_0)\times f(x_1)=-0.3863<0$.

$x_2=\frac{x_0+x_1}{2}=1.5$, $f(x_2)=-0.3918$ $\Rightarrow x_3=\frac{x_1+x_2}{2}=1.75$ ,

$f(x_3)=-0.02061$ $\Rightarrow x_4=\frac{x_3+x_1}{2}=1.875$, $f(x_4)=+0.1786$

$\Rightarrow x_5=\frac{x_3+x_4}{2}=1.8125$ , $f(x_5)=0.07791$

$\vdots$

$x_6=1.78125$ $x_7=1.765625$ $x_8=1.7578125$ $x_9=1.76171875$

$x_{10}=1.763671875$ $x_{11}=1.762953125$

So the root is $x_{11}=1.762953125$ with error $\varepsilon \le 0.001$

H.W:
Use six iteration to locate the root of equation $x^3 - x^2 - 3x - 3 = 0$ by using bisection method

**Theorem:**

Let $f \in C[a,b]$ and suppose $f(a)\times f(b)<0$. The bisection procedure generates a sequence $\{P_n\}$ approximating P with the property $|P_n - P| \le \frac{b-a}{2^n};\ n \ge 1$.

**Proof:** For each $n \ge 1$ we have $b_n - a_n = \frac{b-a}{2^{n-1}}$ and $P \in (a_n, b_n)$. Since $P_n = \frac{1}{2}(a+b)$ for all $n \ge 1$, it follows that $|P_n - P| \le \frac{1}{2}(b_n - a_n) = \frac{b-a}{2^n};$ for all $n \ge 1$.

**Example 3:**

Determine approximately how many iterations are necessary to solve f(x) with error $\le \varepsilon$ over [a,b].

**Solution:** We must find an integer n that will satisfy $|P_n - P| \le \frac{b-a}{2^n} \le \varepsilon \Rightarrow 2^n \ge \frac{b-a}{\varepsilon}$

$$\Rightarrow n\ln(2)\,) \ge \ln\left(\frac{b-a}{\varepsilon}\right) \quad \Rightarrow n \ge \frac{\ln\left(\frac{b-a}{\varepsilon}\right)}{\ln(2)}.$$

If $f(x)=x^3+4x-10=0$, $\varepsilon = 10^{-5}$ and [1,2].

$$n \ge \frac{\ln\left(\frac{2-1}{10^{-5}}\right)}{\ln(2)} = \frac{\ln(10^5)}{\ln(2)} = \frac{11.512925}{0.693147} \approx 16.6096 \Rightarrow \text{n=17(number of iterations)}.$$