TECHNIQUES OF COUNTING

FACTORIAL NOTATION

The product of the positive integers from 1 to n inclusive occurs very often in mathematics and hence is denoted by the special symbol $n!$ (read "n factorial"):

$$n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-2)(n-1)n$$

It is also convenient to define $0! = 1$.

Example : $2! = 1 \cdot 2 = 2, \quad 3! = 1 \cdot 2 \cdot 3 = 6, \quad 4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24,$
$5! = 5 \cdot 4! = 5 \cdot 24 = 120, \quad 6! = 6 \cdot 5! = 6 \cdot 120 = 720$

Example : $\frac{8!}{6!} = \frac{8 \cdot 7 \cdot 6!}{6!} = 8 \cdot 7 = 56 \qquad 12 \cdot 11 \cdot 10 = \frac{12 \cdot 11 \cdot 10 \cdot 9!}{9!} = \frac{12!}{9!}$

PERMUTATIONS

An arrangement of a set of n objects in a given order is called a *permutation* of the objects (taken all at a time). An arrangement of any $r \leq n$ of these objects in a given order is called an *r-permutation* or a *permutation of the n objects taken r at a time.*

Example : Consider the set of letters a, b, c and d. Then:
(i) *bdca*, *dcba* and *acdb* are permutations of the 4 letters (taken all at a time);
(ii) *bad*, *adb*, *cbd* and *bca* are permutations of the 4 letters taken 3 at a time;
(iii) *ad*, *cb*, *da* and *bd* are permutations of the 4 letters taken 2 at a time.

The number of permutations of n objects taken r at a time will be denoted by

$$P(n, r)$$

Before we derive the general formula for $P(n, r)$ we consider a special case.

Theorem : $P(n,r) = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}$

The second part of the formula follows from the fact that

$$n(n-1)(n-2)\cdots(n-r+1) = \frac{n(n-1)(n-2)\cdots(n-r+1)\cdot(n-r)!}{(n-r)!} = \frac{n!}{(n-r)!}$$

In the special case that $r = n$, we have

$$P(n,n) = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1 = n!$$

Namely,

Corollary : There are $n!$ permutations of n objects (taken all at a time).

Example : How many permutations are there of 3 objects, say, a, b and c?

By the above corollary there are $3! = 1 \cdot 2 \cdot 3 = 6$ such permutations. These are abc, acb, bac, bca, cab, cba.

Theorem (Binomial Theorem):

$$\begin{aligned}(a+b)^n &= \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r \\ &= a^n + na^{n-1}b + \frac{n(n-1)}{1 \cdot 2} a^{n-2}b^2 + \cdots + nab^{n-1} + b^n\end{aligned}$$

Example :

$$\begin{aligned}(a+b)^5 &= a^5 + 5a^4b + \frac{5 \cdot 4}{1 \cdot 2} a^3b^2 + \frac{5 \cdot 4}{1 \cdot 2} a^2b^3 + 5ab^4 + b^5 \\ &= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5\end{aligned}$$

$$\begin{aligned}(a+b)^6 &= a^6 + 6a^5b + \frac{6 \cdot 5}{1 \cdot 2} a^4b^2 + \frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3} a^3b^3 + \frac{6 \cdot 5}{1 \cdot 2} a^2b^4 + 6ab^5 + b^6 \\ &= a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6\end{aligned}$$

COMBINATIONS

Suppose we have a collection of n objects. A *combination* of these n objects *taken r at a time*, or an *r-combination*, is any subset of r elements. In other words, an r-combination is any selection of r of the n objects

$$C(n, r) = \frac{P(n, r)}{r!} = \frac{n!}{r!\,(n-r)!}$$

Example : How many committees of 3 can be formed from 8 people? Each committee is essentially a combination of the 8 people taken 3 at a time. Thus

$$C(8,3) = \binom{8}{3} = \frac{8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3} = 56$$

Theorem : Let A contain n elements and let $n_1, n_2, \ldots, n_r$ be positive integers with $n_1 + n_2 + \cdots + n_r = n$. Then there exist

$$\frac{n!}{n_1!\, n_2!\, n_3! \cdots n_r!}$$

different ordered partitions of A of the form $(A_1, A_2, \ldots, A_r)$ where A_1 contains n_1 elements, A_2 contains n_2 elements, $\ldots$, and A_r contains n_r elements.

Example : In how many ways can 9 toys be divided between 4 children if the youngest child is to receive 3 toys and each of the other children 2 toys?

We wish to find the number of ordered partitions of the 9 toys into 4 cells containing 3, 2, 2 and 2 toys respectively. By the above theorem, there are

$$\frac{9!}{3!\,2!\,2!\,2!} = 7560$$

such ordered partitions.

Solved problem

1. Find n if (i) $P(n,2) = 72$, (ii) $P(n,4) = 42P(n,2)$, (iii) $2P(n,2) + 50 = P(2n,2)$.

(i) $P(n,2) = n(n-1) = n^2 - n$; hence $n^2 - n = 72$ or $n^2 - n - 72 = 0$ or $(n-9)(n+8) = 0$.

Since n must be positive, the only answer is $n = 9$.

2.

(i) $\binom{16}{3} = \dfrac{16 \cdot 15 \cdot 14}{1 \cdot 2 \cdot 3} = 560$

(ii) $\binom{12}{4} = \dfrac{12 \cdot 11 \cdot 10 \cdot 9}{1 \cdot 2 \cdot 3 \cdot 4} = 495$

3.

Expand and simplify: $(2x + y^2)^5$.

$$\begin{aligned}(2x + y^2)^5 &= (2x)^5 + \frac{5}{1}(2x)^4(y^2) + \frac{5 \cdot 4}{1 \cdot 2}(2x)^3(y^2)^2 + \frac{5 \cdot 4}{1 \cdot 2}(2x)^2(y^2)^3 + \frac{5}{1}(2x)(y^2)^4 + (y^2)^5 \\ &= 32x^5 + 80x^4y^2 + 80x^3y^4 + 40x^2y^6 + 10xy^8 + y^{10}\end{aligned}$$

4.

In how many ways can a committee consisting of 3 men and 2 women be chosen from 7 men and 5 women?

The 3 men can be chosen from the 7 men in $\binom{7}{3}$ ways, and the 2 women can be chosen from the 5 women in $\binom{5}{2}$ ways. Hence the committee can be chosen in $\binom{7}{3}\binom{5}{2} = \dfrac{7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3} \cdot \dfrac{5 \cdot 4}{1 \cdot 2} = 350$ ways.

5.

A student is to answer 8 out of 10 questions on an exam. (i) How many choices has he? (ii) How many if he must answer the first 3 questions? (iii) How many if he must answer at least 4 of the first 5 questions?

(i) The 8 questions can be selected in $\binom{10}{8} = \binom{10}{2} = \dfrac{10 \cdot 9}{1 \cdot 2} = 45$ ways.

(ii) If he answers the first 3 questions, then he can choose the other 5 questions from the last 7 questions in $\binom{7}{5} = \binom{7}{2} = \dfrac{7 \cdot 6}{1 \cdot 2} = 21$ ways.

(iii) If he answers all the first 5 questions, then he can choose the other 3 questions from the last 5 in $\binom{5}{3} = 10$ ways. On the other hand, if he answers only 4 of the first 5 questions, then he can choose these 4 in $\binom{5}{4} = \binom{5}{1} = 5$ ways, and he can choose the other 4 questions from the last 5 in $\binom{5}{4} = \binom{5}{1} = 5$ ways; hence he can choose the 8 questions in $5 \cdot 5 = 25$ ways. Thus he has a total of 35 choices.