Real functions

A function is a relation that uniquely associates members of one set with members of another set. A function whose range is in the real numbers $\mathbb{R}$ is said to be a real function, also called a real-valued function $f : X \to \mathbb{R}$; $X \subseteq \mathbb{R}$.

The domain of a function is the set of all the numbers you can substitute in to the function (x - values).

The range of a function is the set of all the numbers you can get out of the function (y - values).

When determining the domain of a function from a formula, we really only have to look out for two situations:

1- Rational expression (fractions) – Division by zero is not allowed so we must omit any values of x which make the denominator zero.

2- Even roots – For even roots such as square roots, the radicand cannot be negative. That is the radicand greater than or equal to zero.

Examples: Find the domain for the functions

1. $f(x) = \dfrac{5x}{x^2 - 3x - 4}$

Since this is a rational expression, we must not let the denominator equal zero.

What values of x make denominator equal zero?

$$x^2 - 3x - 4 = 0 \;\Rightarrow\; (x-4)(x+1) = 0 \;\Rightarrow\; x = 4 \text{ or } x = -3$$

Thus the domain is $D = \mathbb{R}/\{4, -3\}$

2. $f(x) = \sqrt{2x+5} - 13$

Since this function involves a square root we must make sure the radicand is non-negative

$$2x + 5 \geq 0 \;\Rightarrow\; x \geq -\frac{5}{2}$$

Thus the domain is

$$D = \left[-\frac{5}{2}, \infty\right)$$

3. $f(x) = \dfrac{2x}{\sqrt{3x-4}}$

We must be very careful with this function since it involves both rational expression and square root. The square root requires the radicand be greater than or equal to zero, that is, $3x-4 \geq 0$. However , since the square root is in the denominator and we cannot divide by zero.

Thus $3x-4>0 \quad \Rightarrow \quad 3x>4$, then the domain is

$$D = \left(\frac{3}{4}, \infty\right)$$

4. $f(x) = \dfrac{\sqrt{x+5}}{x-2}$

$x+5 \geq 0 \quad \Rightarrow \quad x \geq -5$

And $\quad x-2 \neq 0 \quad \Rightarrow \quad x \neq 2$

Thus the domain is $D = [-5,2) \cup (2,\infty)$

5. $f(x) = \sqrt{x^2+8x+15}$

$x^2+8x+15 \geq 0$

We will solve the related quadratic equation $\quad x^2+8x+15=0$

$$(x+5)(x+3)=0 \quad \Rightarrow \quad x=-5\,,-3$$

These two critical numbers will separate the number-line into three intervals

$$(-\infty,-5]\,,[-5,-3] \text{ and } [-3,\infty)$$

If we choose $-6 \in (-\infty,-5] \;\Rightarrow\; (-6)^2+8\times(-6)+15 \geq 0 \;\Rightarrow\; 3 \geq 0$ True √

If we choose $-4 \in [-5,-3] \;\Rightarrow\; (-4)^2+8\times(-4)+15 \geq 0 \Rightarrow\; -1 \geq 0$ False ×

If we choose $0 \in [-3,\infty) \;\Rightarrow\; (0)^2+8\times 0+15 \geq 0 \;\Rightarrow\; 15 \geq 0$ True √

Thus the domain is $D = (-\infty,-5] \cup [-3,\infty)$

6. $f(x) = \dfrac{1}{\sqrt{x^2+8x+15}}$

Since the square root is in the denominator and we cannot divide by zero.

$x^2+8x+15>0 \quad \Rightarrow \quad D = (-\infty,-5) \cup (-3,\infty)$

$$7.\ f(x) = \sqrt{\frac{6-x}{x+5}}$$

$$\frac{6-x}{x+5} \geq 0 \qquad \text{such that } x \neq -5$$

The zero of numerator is $6 - x = 0 \Rightarrow x = 6$

The zero of denominator is $x + 5 = 0 \Rightarrow x = -5$

These two critical numbers will separate the number-line into three intervals

$$(-\infty, -5)\ , \qquad (-5, 6] \text{ and } [6, \infty)$$

If we choose

$$-6 \in (-\infty, -5) \Rightarrow \frac{6-x}{x+5} \geq 0 \Rightarrow \frac{6+6}{-6+1} \geq 0 \Rightarrow \frac{12}{-5} \geq 0 \quad \text{False} \quad \times$$

If we choose

$$0 \in (-5, 6] \Rightarrow \frac{6-x}{x+5} \geq 0 \Rightarrow \frac{6-0}{0+5} \geq 0 \Rightarrow \frac{6}{5} \geq 0 \quad \text{True} \quad \surd$$

If we choose

$$7 \in [6, \infty) \Rightarrow \frac{6-x}{x+5} \geq 0 \Rightarrow \frac{6-7}{7+5} \geq 0 \Rightarrow \frac{-1}{12} \geq 0 \quad \text{False} \quad \times$$

Thus the domain is $D = (-5, 6]$

Exercises

Find the domain for the functions

1. $f(x) = 8 - \sqrt{3x - 7}$
2. $f(x) = \sqrt{x^2 - 6x + 8}$
3. $f(x) = \dfrac{x+3}{x^2 - 2x - 8}$
4. $f(x) = \dfrac{4x}{\sqrt{x^2 + 2x - 15}}$
5. $f(x) = \dfrac{\sqrt{2x+5}}{2-x}$
6. $f(x) = \sqrt{\dfrac{x-2}{x+4}}$