Second -order differential equations

1-Homogeneous linear equations with constant coefficients

A second order homogeneous equation with constant coefficients is written as

$$ay'' + by' + cy = 0 \quad \text{where } a,\ b \text{ and } c \text{ are constant.}$$

Let us summarize the steps to follow in order to find the general solution:

1. Write down the **characteristic equation** $am^2 + bm + c = 0$

2. Find the roots of the characteristic equation m_1 and m_2

Here we have three cases

I. If m_1 and m_2 are distinct real numbers $(m_1 = m_2)$ (this happens if $b^2 - 4ac > 0$), then the general solution is

$$y_h = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

II. If $m_1 = m_2$ (this happens if $b^2 - 4ac = 0$), then the general solution is

$$y_h = (c_1 x + c_2) e^{mx}$$

III. If m_1 and m_2 are conjugate complex numbers $(m_1 = \overline{m_2} = \alpha + \beta i)$ (this happens if $b^2 - 4ac < 0$), then the general solution is

$$y_h = e^{\alpha x}(c_1 \sin \beta x + c_2 \cos \beta x)$$

Example 1: Find the general solution of $y'' - 6y' + 8y = 0$

Solution

Characteristic equation and its roots

$$m^2 - 6m + 8 = 0 \ \Rightarrow\ (m-2)(m-4) = 0 \ \Rightarrow\ m_1 = 2 \text{ and } m_2 = 4$$ □

The general solution is

$$y_h = c_1 e^{2x} + c_2 e^{4x}$$ □

Example 2: Find the general solution of $y'' + y' - 20y = 0$

Solution

$$m^2 + m - 20 = 0 \quad \text{(Characteristic equation)}$$ □

$$(m+4)(m-5) = 0 \ \Rightarrow\ m_1 = -4 \text{ and } m_2 = 5 \quad \text{(The roots)}$$ □

$$y_h = c_1 e^{-4x} + c_2 e^{5x} \quad \text{(General solution)}$$

Example 3: Find the general solution of $y'' - 6y' + 9y = 0$

Solution

$$m^2 - 6m + 9 = 0$$

$$(m-3)(m-3) = 0 \quad \Rightarrow \quad m_1 = m_2 = 3$$

$$y_h = (c_1 x + c_2)e^{3x}$$

Example 4: Find the general solution of $y'' - 2y' + 5y = 0$

Solution

$$m^2 - 2m + 5 = 0$$

$$m = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a} = \frac{-(-2) \mp \sqrt{(-2)^2 - 4 \times 1 \times 5}}{2 \times 1} = 1 \mp 2i$$

$$y_h = e^{x}(c_1 \sin 2x + c_2 \cos 2x)$$

Example 5: Solve $y'' + 2y' + 10y = 0$ with $y'(0) = 9$ and $y(0) = 3$

Solution $m^2 + 2m + 10 = 0$

$$m = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \mp \sqrt{(2)^2 - 4 \times 1 \times 10}}{2 \times 1} = -1 \mp 3i$$

$$y_h = e^{-x}[c_1 \sin 3x + c_2 \cos 3x]$$ (General solution)

$$y(0) = 3 \quad \Rightarrow 3 = e^{0}[c_1 \sin 0 + c_2 \cos 0] \quad \Rightarrow \quad c_2 = 3$$

$$y_h = e^{-x}[c_1 \sin 3x + 3 \cos 3x]$$

$$y_h' = e^{-x}[3c_1 \cos 3x - 9 \sin 3x] - e^{-x}[c_1 \sin 3x + 3 \cos 3x]$$

$$y'(0) = 9 \quad \Rightarrow 9 = e^{0}[3c_1 \cos 0 - 9 \sin 0] - e^{0}[c_1 \sin 0 + 3 \cos 0]$$

$$9 = 3c_1 - 3 \Rightarrow \quad c_1 = 4$$

$$y_h = e^{-x}[4 \sin 3x + 3 \cos 3x]$$ (Particular solution)

Example 6: Solve $y'' + 9y = 0$ with $y'(\pi/6) = 3$ and $y(\pi/6) = 2$

Solution $m^2 + 9 = 0 \quad \Rightarrow \quad m = \mp 3i$

$$y_h = c_1 \sin 3x + c_2 \cos 3x$$ (General solution)

$$y(\pi/6) = 2 \quad \Rightarrow \quad 2 = c_1 \sin(\pi/2) + c_2 \cos(\pi/2) \quad \Rightarrow c_1 = 2$$

$$y_h' = 3c_1 \cos 3x - 3c_2 \sin 3x$$

$$y'(\pi/6) = 3 \quad \Rightarrow \quad c_2 = -1$$

$$y_h = 2 \sin 3x - \cos 3x$$ (Particular solution)

Example 7: Solve $y'' + 9y' + 14y = 0$ with $y'(5\pi) = 2$ and $y(5\pi) = 4$

$$m^2 + 9m + 14 = 0$$

$$(m+2)(m+7) = 0 \quad \Rightarrow \quad m_1 = -2 \ and \ m_2 = -7$$

$$y_h = c_1e^{-2x} + c_2e^{-7x} \quad \text{(General solution)}$$

$$y(5\pi) = 4 \quad \Rightarrow \quad 4 = c_1e^{-10\pi} + c_2e^{-35\pi} \quad \cdots \quad eq(1)$$

$$y_h' = -2c_1e^{-2x} - 7c_2e^{-7x}$$

$$y'(5\pi) = 2 \quad \Rightarrow \quad 2 = -2c_1e^{-10\pi} - 7c_2e^{-35\pi} \quad \cdots \quad eq(2)$$

$$eq(1) \times 2 \quad \Rightarrow \quad 8 = 2c_1e^{-10\pi} + 2c_2e^{-35\pi}$$

Then $$10 = -5c_2e^{-35\pi}$$

$$\boxed{c_2 = -2e^{35\pi}} \quad \cdots \quad eq(3)$$

For $eq(3)$ and $eq(1)$ we get $4 = c_1e^{-10\pi} - 2 \quad \Rightarrow \quad \boxed{c_1 = 6e^{10\pi}}$

$$y_h = 6e^{10\pi}e^{-2x} - 2e^{35\pi}e^{-7x}$$

$$y_h = 6e^{10\pi-2x} - 2e^{35\pi-7x} \quad \text{(Particular solution)}$$

Exercises

For each of the following problem (1 through 10), find the general solution of the differential equation

(1) $y'' + y' - 2y = 0$ (2) $y'' - 4y' + 4y = 0$

(3) $y'' + 2y' + 2y = 0$ (4) $y'' + 2y' = 0$

(5) $y'' + 6y' + 5y = 0$ (6) $y'' + 2y' + 4y = 0$

(7) $y'' + 9y' + 8y = 0$ (8) $y'' - 6y' + 25y = 0$

(9) $y'' - 6y' + 9y = 0$ (10) $y'' - 4y' + 13y = 0$

For each of the following problem (11 through 14),solve the initial-value problem

(11) $y'' + 5y' + 4y = 0$; $y'(0) = -7$ and $y(0) = 1$

(12) $y'' + y' - 12y = 0$; $y'(\pi) = -20$ and $y(\pi) = -2$

(13) $9y'' + y = 0$; $y'(0) = 2$ and $y(0) = -2$

(14) $y'' + 2y' - 4y = 0$; $y'(0) = -6$ and $y(0) = 6$