Dimensional Analysis
and Similarity

Introduction

Motivation. In this chapter we discuss the planning, presentation, and interpretation of experimental data. We shall try to convince you that such data are best presented in *dimensionless* form. Experiments which might result in tables of output, or even multiple volumes of tables, might be reduced to a single set of curves—or even a single curve—when suitably nondimensionalized. The technique for doing this is *dimensional analysis*. In the last lecture presented gross control-volume balances of mass, momentum, and energy which led to estimates of global parameters: mass flow, force, torque, total heat transfer. In the last lecture presented infinitesimal balances which led to the basic partial differential equations of fluid flow and some particular solutions. These two chapters covered *analytical* techniques, which are limited to fairly simple geometries and well-defined boundary conditions. Probably one-third of fluid-flow problems can be attacked in this analytical or theoretical manner.

The other two-thirds of all fluid problems are too complex, both geometrically and physically, to be solved analytically. They must be tested by experiment. Their behavior is reported as experimental data. Such data are much more useful if they are expressed in compact, economic form. Graphs are especially useful, since tabulated data cannot be absorbed, nor can the trends and rates of change be observed, by most engineering eyes. These are the motivations for dimensional analysis. The technique is traditional in fluid mechanics and is useful in all engineering and physical sciences, with notable uses also seen in the biological and social sciences. Dimensional analysis can also be useful in theories, as a compact way to present an analytical solution or output from a computer model. Here we concentrate on the presentation of experimental fluid-mechanics data.

Basically, dimensional analysis is a method for reducing the number and complexity of experimental variables which affect a given physical phenomenon, by using a sort of compacting technique. If a phenomenon depends upon n dimensional variables, dimensional analysis will reduce the problem to only k dimensionless variables, where the reduction $n - k = 1, 2, 3$, or 4, depending upon the problem complexity. Generally $n - k$ equals the number of different dimensions (sometimes called basic or primary or fundamental dimensions) which govern the problem. In fluid mechanics, the four basic dimensions are usually taken to be mass M, length L, time T, and temperature θor an $MLT\theta$ system for short. Sometimes one uses an $FLT\theta$

system, with force F replacing mass. Although its purpose is to reduce variables and group them in dimensionless form, dimensional analysis has several side benefits. The first is enormous savings in time and money. Suppose one knew that the force F on a particular body immersed in a stream of fluid depended only on the body length L, stream velocity V, fluid density ρ, and fluid viscosity μ, that is,

$$F = f(L, V, \rho, \mu)$$

The Principle of Dimensional Homogeneity

Historically, the first person to write extensively about units and dimensional reasoning in physical relations was Euler in 1765. Euler's ideas were far ahead of his time, as were those of Joseph Fourier, whose 1822 book Analytical Theory of Heat outlined what is now called the principle of dimensional homogeneity and even developed some similarity rules for heat flow. There were no further significant advances until Lord Rayleigh's book in 1877, Theory of Sound, which proposed a "method of dimensions" and gave several examples of dimensional analysis. The final breakthrough which established the method as we know it today is generally credited to E. Buckingham in 1914

(If an equation truly expresses a proper relationship between variables in a physical process, it will be *dimensionally homogeneous;* i.e., each of its additive terms will have the same dimensions).

The Pi Theorem

There are several methods of reducing a number of dimensional variables into a smaller number of dimensionless groups. The scheme given here was proposed in 1914 by Buckingham and is now called the Buckingham pi theorem. The name pi comes from the mathematical notation _, meaning a product of variables. The dimensionless groups found from the theorem are power products denoted by II1, II2, II3, etc. The method allows the pis to be found in sequential order without resorting to free exponents. The first part of the pi theorem explains what reduction in variables to expect:

If a physical process satisfies the PDH and involves n dimensional variables, it can be reduced to a relation between only k dimensionless variables or II's. The reduction j = n - k equals the maximum number of variables which do not form a pi among themselves and is always less than or equal to the number of dimensions describing the variables.

Typically, six steps are involved:

1. List and count the n variables involved in the problem. If any important variables are missing, dimensional analysis will fail.
2. List the dimensions of each variable according to {MLTθ} or {FLTθ}. A list is given in Table.
3. Find j. Initially guess j equal to the number of different dimensions present, and look for j variables which do not form a pi product. If no luck, reduce j by 1 and look again. With practice, you will find j rapidly.
4. Select j scaling parameters which do not form a pi product. Make sure they please you and have some generality if possible, because they will then appear in every one of your pi groups. Pick density or velocity or length. Do not pick surface tension, e.g., or you will form six different independent Weber-number parameters and thoroughly annoy your colleagues.
5. Add one additional variable to your j repeating variables, and form a power product. Algebraically find the exponents which make the product dimensionless. Try to arrange for your output or dependent variables (force, pressure drop, torque, power) to appear in the numerator, and your plots will look better. D this sequentially, adding one new variable each time, and you will find all n -j = k desired pi products.
6. Write the final dimensionless function, and check your work to make sure all pi groups are dimensionless.