3 CONTINUOUS DISTRIBUTIONS

In this section several parametric families of univariate probability density functions are presented. Sketches of some are included; the mean and variance (when they exist) of each are given.

3.1 Uniform or Rectangular Distribution

A very simple distribution for a continuous random variable is the uniform distribution. It is particularly useful in theoretical statistics because it is convenient to deal with mathematically.

Definition 10 Uniform distribution If the probability density function of a random variable X is given by

$$f_X(x) = f_X(x; a, b) = \frac{1}{b-a} I_{[a,b]}(x), \tag{21}$$

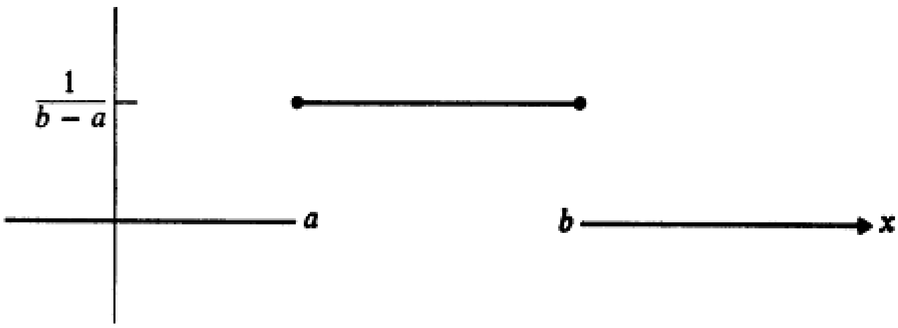

FIGURE 8
Uniform probability density.

where the parameters a and b satisfy $-\infty < a < b < \infty$, then the random variable X is defined to be *uniformly* distributed over the interval $[a, b]$, and the distribution given by Eq. (21) is called a *uniform distribution.*

Theorem 12 If X is uniformly distributed over $[a, b]$, then

$$\mathscr{E}[X] = \frac{a+b}{2}, \qquad \operatorname{var}[X] = \frac{(b-a)^2}{12}, \qquad \text{and} \qquad m_X(t) = \frac{e^{bt} - e^{at}}{(b-a)t}. \tag{22}$$

PROOF

$$\mathscr{E}[X] = \int_a^b x \frac{1}{b-a}\, dx = \frac{b^2 - a^2}{2(b-a)} = \frac{a+b}{2}.$$

$$\operatorname{var}[X] = \mathscr{E}[X^2] - (\mathscr{E}[X])^2 = \int_a^b x^2 \frac{1}{b-a}\, dx - \left(\frac{a+b}{2}\right)^2$$

$$= \frac{b^3 - a^3}{3(b-a)} - \frac{(a+b)^2}{4} = \frac{(b-a)^2}{12}.$$

$$m_X(t) = \mathscr{E}[e^{tX}] = \int_a^b e^{tx} \frac{1}{b-a}\, dx = \frac{e^{bt} - e^{at}}{(b-a)t}.$$ ////

The uniform distribution gets its name from the fact that its density is uniform, or constant, over the interval $[a, b]$. It is also called the *rectangular* distribution—the shape of the density is rectangular.

The cumulative distribution function of a uniform random variable is given by

$$F_X(x) = \left(\frac{x-a}{b-a}\right) I_{[a,b]}(x) + I_{(b,\infty)}(x). \tag{23}$$

3.2 Normal Distribution

A great many of the techniques used in applied statistics are based upon the normal distribution; it will frequently appear in the remainder of this book.

Definition 11 Normal distribution A random variable X is defined to be *normally* distributed if its density is given by

$$f_X(x) = f_X(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}, \qquad (24)$$

where the parameters μ and σ satisfy $-\infty < \mu < \infty$ and $\sigma > 0$. Any distribution defined by a density function given in Eq. (24) is called a *normal distribution.* ////

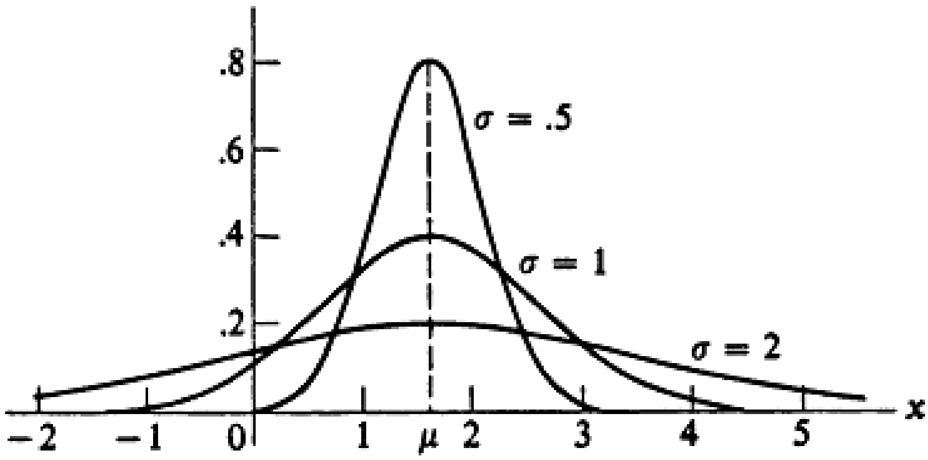

FIGURE 9
Normal densities.

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \quad \text{and} \quad \Phi(x) = \int_{-\infty}^{x} \phi(u)\, du. \qquad (25)$$

Since $\phi_{\mu,\sigma^2}(x)$ is given to be a density function, it is implied that

$$\int_{-\infty}^{\infty} \phi_{\mu,\sigma^2}(x)\, dx = 1,$$

but we should satisfy ourselves that this is true. The verification is somewhat troublesome because the indefinite integral of this particular density function does not have a simple functional expression. Suppose that we represent the area under the curve by A; then

$$A = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-(x-\mu)^2/2\sigma^2}\, dx,$$

and on making the substitution $y = (x - \mu)/\sigma$, we find that

$$A = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2}\, dy.$$

We wish to show that $A = 1$, and this is most easily done by showing that A^2 is 1 and then reasoning that $A = 1$ since $\phi_{\mu,\sigma^2}(x)$ is positive. We may put

$$A^2 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2}\, dy \, \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2}\, dz$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(y^2+z^2)}\, dy\, dz$$

by writing the product of two integrals as a double integral. In this integral we change the variables to polar coordinates by the substitutions

$$y = r \sin \theta$$
$$z = r \cos \theta,$$

and the integral becomes

$$A^2 = \frac{1}{2\pi} \int_0^{\infty} \int_0^{2\pi} re^{-\frac{1}{2}r^2}\, d\theta\, dr$$
$$= \int_0^{\infty} re^{-\frac{1}{2}r^2}\, dr$$
$$= 1.$$

Theorem 13 If X is a normal random variable,

$$\mathscr{E}[X] = \mu, \qquad \text{var}\,[X] = \sigma^2, \qquad \text{and} \qquad m_X(t) = e^{\mu t + \sigma^2 t^2/2}. \tag{26}$$

PROOF

$$m_X(t) = \mathscr{E}[e^{tX}] = e^{t\mu}\mathscr{E}[e^{t(X-\mu)}]$$
$$= e^{t\mu} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{t(x-\mu)} e^{-(1/2\sigma^2)(x-\mu)^2}\, dx$$
$$= e^{t\mu} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(1/2\sigma^2)[(x-\mu)^2 - 2\sigma^2 t(x-\mu)]}\, dx.$$

If we complete the square inside the bracket, it becomes

$$(x-\mu)^2 - 2\sigma^2 t(x-\mu) = (x-\mu)^2 - 2\sigma^2 t(x-\mu) + \sigma^4 t^2 - \sigma^4 t^2$$
$$= (x - \mu - \sigma^2 t)^2 - \sigma^4 t^2,$$

and we have

$$m_X(t) = e^{t\mu} e^{\sigma^2 t^2/2} \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-(x-\mu-\sigma^2 t)^2/2\sigma^2}\, dx.$$

The integral together with the factor $1/\sqrt{2\pi}\sigma$ is necessarily 1 since it is the area under a normal distribution with mean $\mu + \sigma^2 t$ and variance σ^2. Hence,

$$m_X(t) = e^{\mu t + \sigma^2 t^2/2}.$$

On differentiating $m_X(t)$ twice and substituting $t = 0$, we find

$$\mathscr{E}[X] = m_X'(0) = \mu$$

and

$$\text{var}\,[X] = \mathscr{E}[X^2] - (\mathscr{E}[X])^2 = m_X''(0) - \mu^2 = \sigma^2,$$

thus justifying our use of the symbols μ and σ^2 for the parameters. ////

Since the indefinite integral of $\phi_{\mu,\sigma^2}(x)$ does not have a simple functional form, one can only exhibit the cumulative distribution function as

$$\Phi_{\mu,\sigma^2}(x) = \int_{-\infty}^{x} \phi_{\mu,\sigma^2}(u)\, du. \qquad (27)$$

The following theorem shows that we can find the probability that a normally distributed random variable, with mean μ and variance σ^2, falls in any interval in terms of the standard normal cumulative distribution function, and this standard normal cumulative distribution function is tabled in Table 2 of Appendix D.

Theorem 14 If $X \sim N(\mu, \sigma^2)$, then

$$P[a < X < b] = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right). \qquad (28)$$

PROOF

$$\begin{aligned} P[a < X < b] &= \int_a^b \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}[(x-\mu)/\sigma]^2}\, dx \\ &= \int_{(a-\mu)/\sigma}^{(b-\mu)/\sigma} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\, dz \\ &= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right). \end{aligned}$$ ////

3.3 Exponential and Gamma Distributions

Two other families of distributions that play important roles in statistics are the (negative) exponential and gamma distributions, which are defined in this subsection. The reason that the two are considered together is twofold; first, the nential is a special case of the gamma, and, second, the sum of independent :ically distributed exponential random variables is gamma-distributed, as hall see in Chap. V.

Definition 12 Exponential distribution If a random variable X has a density given by

$$f_X(x; \lambda) = \lambda e^{-\lambda x} I_{[0, \infty)}(x), \tag{29}$$

where $\lambda > 0$, then X is defined to have an (negative) *exponential distribution.* ////

Definition 13 Gamma distribution If a random variable X has density given by

$$f_X(x; r, \lambda) = \frac{\lambda}{\Gamma(r)} (\lambda x)^{r-1} e^{-\lambda x} I_{[0, \infty)}(x), \tag{30}$$

where $r > 0$ and $\lambda > 0$, then X is defined to have a *gamma distribution.* $\Gamma(\cdot)$ is the gamma function and it is discussed in Appendix A. ////

Remark If in the gamma density $r = 1$, the gamma density specializes to the exponential density. ////

Theorem 15 If X has an exponential distribution, then

$$\mathscr{E}[X] = \frac{1}{\lambda}, \qquad \text{var}\,[X] = \frac{1}{\lambda^2}, \qquad \text{and} \qquad m_X(t) = \frac{\lambda}{\lambda - t} \qquad \text{for} \qquad t < \lambda. \tag{31}$$

PROOF The exponential distribution was the distribution used as an example for some definitions given in Chap. II, and derivations of the above appear there. Also, Theorem 15 is a corollary to the following theorem.

Theorem 16 If X has a gamma distribution with parameters r and λ, then

$$\mathscr{E}[X] = \frac{r}{\lambda}, \qquad \text{var}\,[X] = \frac{r}{\lambda^2}, \qquad \text{and} \qquad m_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^r \qquad \text{for } t < \lambda. \tag{32}$$

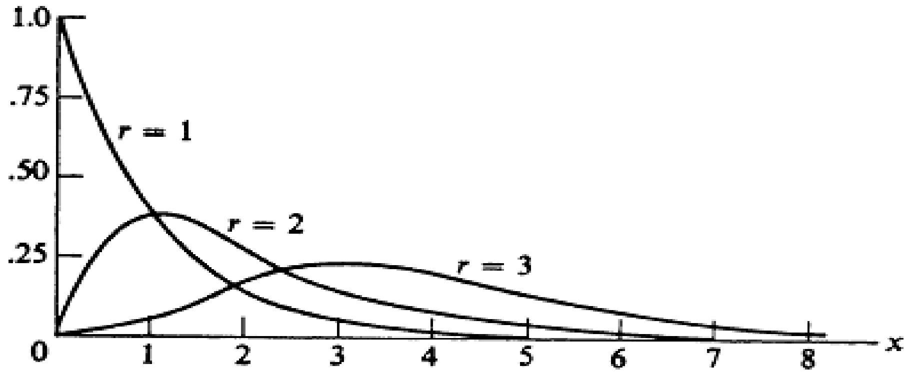

FIGURE 11
Gamma densities ($\lambda = 1$).

PROOF

$$\begin{aligned} m_X(t) &= \mathscr{E}[e^{tX}] \\ &= \int_0^\infty \frac{\lambda^r}{\Gamma(r)} e^{tx} x^{r-1} e^{-\lambda x}\, dx \\ &= \left(\frac{\lambda}{\lambda - t}\right)^r \int_0^\infty \frac{(\lambda - t)^r}{\Gamma(r)} x^{r-1} e^{-(\lambda - t)x}\, dx = \left(\frac{\lambda}{\lambda - t}\right)^r. \\ m_X'(t) &= r\lambda^r(\lambda - t)^{-r-1} \end{aligned}$$

and

$$m_X''(t) = r(r+1)\lambda^r(\lambda - t)^{-r-2};$$

hence

$$\mathscr{E}[X] = m_X'(0) = \frac{r}{\lambda}$$

and

$$\begin{aligned} \text{var}\,[X] &= \mathscr{E}[X^2] - (\mathscr{E}[X])^2 \\ &= m_X''(0) - \left(\frac{r}{\lambda}\right)^2 = \frac{r(r+1)}{\lambda^2} - \left(\frac{r}{\lambda}\right)^2 = \frac{r}{\lambda^2}. \end{aligned} \qquad ////$$

The exponential distribution has been used as a model for lifetimes of various things. When we introduced the Poisson distribution, we spoke of certain happenings, for example, particle emissions, occurring in time. The length of the time interval between successive happenings can be shown to have an exponential distribution provided that the number of happenings in a fixed

Theorem 17 If the random variable X has a gamma distribution with parameters r and λ, where r is a positive integer, then

$$F_X(x) = 1 - \sum_{j=0}^{r-1} \frac{e^{-\lambda x}(\lambda x)^j}{j!}. \tag{33}$$

Theorem 18 If the random variable X has an exponential distribution with parameter λ, then

$$P[X > a + b \mid X > a] = P[X > b], \qquad \text{for } a > 0 \text{ and } b > 0.$$

PROOF $$P[X > a + b \mid X > a] = \frac{P[X > a + b]}{P[X > a]} = \frac{e^{-\lambda(a+b)}}{e^{-\lambda a}}$$
$$= e^{-\lambda b} = P[X > b].$$ ////

Let X represent the lifetime of a given component; then, in words, Theorem 18 states that the conditional probability that the component will last $a + b$ time units given that it has lasted a time units is the same as its initial probability of lasting b time units. Another way of saying this is to say that an "old" functioning component has the same lifetime distribution as a "new" functioning component or that the component is not subject to fatigue or to wear.

3.4 Beta Distribution

A family of probability densities of continuous random variables taking on values in the interval (0, 1) is the family of beta distributions.

Definition 14 **Beta distribution** If a random variable X has a density given by

$$f_X(x) = f_X(x; a, b) = \frac{1}{B(a, b)} x^{a-1}(1 - x)^{b-1} I_{(0,1)}(x), \tag{34}$$

where $a > 0$ and $b > 0$, then X is defined to have a *beta distribution*. ////

The function $B(a, b) = \int_0^1 x^{a-1}(1 - x)^{b-1}\,dx$, called the *beta function*, is mentioned briefly in Appendix A.

Remark The beta distribution reduces to the uniform distribution over $(0, 1)$ if $a = b = 1$. ////

Remark The cumulative distribution function of a beta-distributed random variable is

$$F_X(x; a, b) = I_{(0,1)}(x) \int_0^x \frac{1}{\mathrm{B}(a, b)} u^{a-1}(1 - u)^{b-1}\, du + I_{[1,\infty)}(x); \tag{35}$$

it is often called the *incomplete beta* and has been extensively tabulated. ////

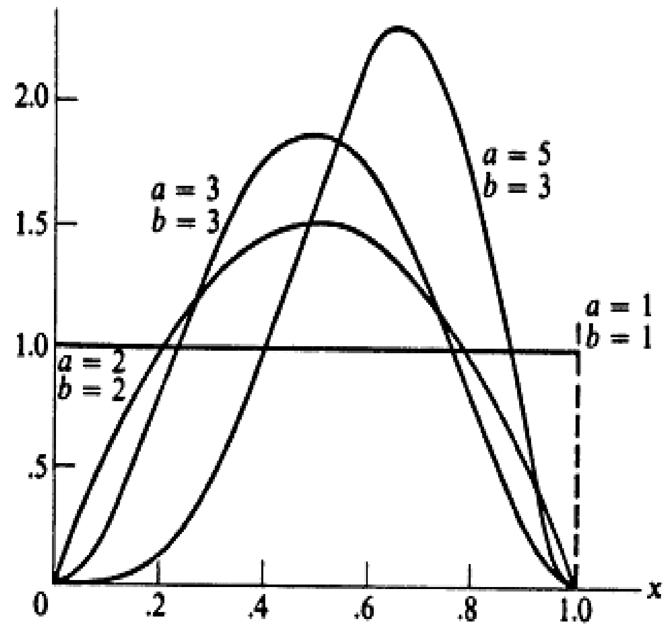

FIGURE 12
Beta densities.

Theorem 19 If X is a beta-distributed random variable, then

$$\mathscr{E}[X] = \frac{a}{a + b} \qquad \text{and} \qquad \operatorname{var}[X] = \frac{ab}{(a + b + 1)(a + b)^2}.$$

PROOF

$$\mathscr{E}[X^k] = \frac{1}{B(a, b)} \int_0^1 x^{k+a-1}(1-x)^{b-1}\, dx$$

$$= \frac{B(k+a, b)}{B(a, b)} = \frac{\Gamma(k+a)\Gamma(b)}{\Gamma(k+a+b)} \cdot \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}$$

$$= \frac{\Gamma(k+a)\Gamma(a+b)}{\Gamma(a)\Gamma(k+a+b)};$$

hence,

$$\mathscr{E}[X] = \frac{\Gamma(a+1)\Gamma(a+b)}{\Gamma(a)\Gamma(a+b+1)} = \frac{a}{a+b},$$

and

$$\operatorname{var}[X] = \mathscr{E}[X^2] - (\mathscr{E}[X])^2 = \frac{\Gamma(a+2)\Gamma(a+b)}{\Gamma(a)\Gamma(a+b+2)} - \left(\frac{a}{a+b}\right)^2$$

$$= \frac{(a+1)a}{(a+b+1)(a+b)} - \left(\frac{a}{a+b}\right)^2 = \frac{ab}{(a+b+1)(a+b)^2}. \qquad ////$$

The family of beta densities is a two-parameter family of densities that is positive on the interval (0, 1) and can assume quite a variety of different shapes, and, consequently, the beta distribution can be used to model an experiment for which one of the shapes is appropriate.