EXAMPLE 5 Let X denote the number of defectives in a sample of size n when sampling is done without replacement from an urn containing M balls, K of which are defective. Then X has a hypergeometric distribution. See Eq. (5) of Subsec. 3.5 in Chap. I. ////

2.4 Poisson Distribution

Definition 5 Poisson distribution A random variable X is defined to have a *Poisson distribution* if the density of X is given by

$$f_X(x) = f_X(x; \lambda) = \begin{cases} \dfrac{e^{-\lambda}\lambda^x}{x!} & \text{for } x = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} = \frac{e^{-\lambda}\lambda^x}{x!} I_{\{0, 1, \ldots\}}(x), \tag{9}$$

where the parameter λ satisfies $\lambda > 0$. The density given in Eq. (9) is called a *Poisson density*. ////

Theorem 6 Let X be a Poisson distributed random variable; then

$$\mathscr{E}[X] = \lambda, \qquad \text{var}[X] = \lambda, \qquad \text{and} \quad m_X(t) = e^{\lambda(e^t - 1)}. \tag{10}$$

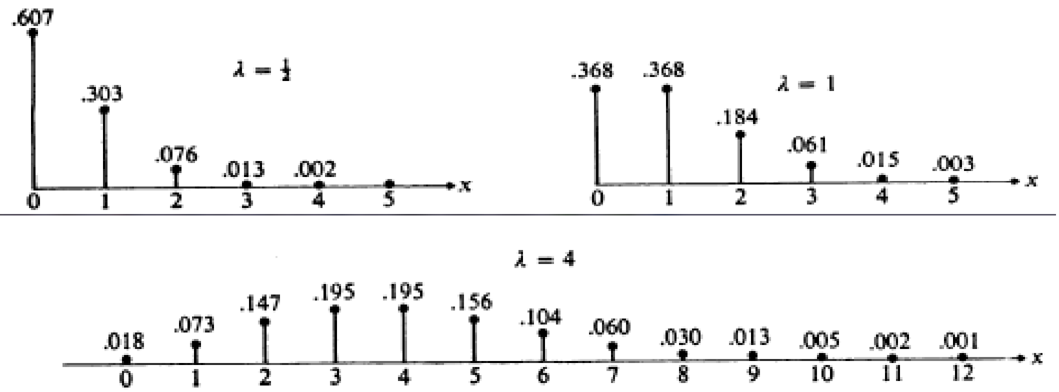

FIGURE 5
Poisson densities.

PROOF

$$m_X(t) = \mathscr{E}[e^{tX}] = \sum_{x=0}^{\infty} \frac{e^{tx}e^{-\lambda}\lambda^x}{x!}$$
$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!} = e^{-\lambda}e^{\lambda e^t};$$

hence,

$$m_X'(t) = \lambda e^{-\lambda}e^t e^{\lambda e^t}$$

and

$$m_X''(t) = \lambda e^{-\lambda}e^t e^{\lambda e^t}[\lambda e^t + 1].$$

So,

$$\mathscr{E}[X] = m_X'(0) = \lambda$$

and

$$\text{var}\,[X] = \mathscr{E}[X^2] - (\mathscr{E}[X])^2 = m_X''(0) - \lambda^2 = \lambda[\lambda + 1] - \lambda^2 = \lambda. \qquad ////$$

The Poisson distribution provides a realistic model for many random phenomena. Since the values of a Poisson random variable are the nonnegative integers, any random phenomenon for which a count of some sort is of interest is a candidate for modeling by assuming a Poisson distribution. Such a count might be the number of fatal traffic accidents per week in a given state, the number of radioactive particle emissions per unit of time, the number of telephone calls per hour coming into the switchboard of a large business, the number of meteorites that collide with a test satellite during a single orbit, the number of organisms per unit volume of some fluid, the number of defects per unit of some material, the number of flaws per unit length of some wire, etc. Naturally, not all counts can be realistically modeled with a Poisson distribution, but some can; in fact, if certain assumptions regarding the phenomenon under observation are satisfied, the Poisson model is the correct model.