Random Experiment

It is an experiment that its outcomes can not be predicted, and each its outcome has the same chance(probability).

Definition 11 Sample space The *sample space*, denoted by Ω, is the collection or totality of all possible outcomes of a conceptual experiment.

Definition 12 Event and event space An *event* is a subset of the sample space. The class of all events associated with a given experiment is defined to be the *event space*. ////

Example

Tossing one single fair coin one time.

Sample space=$\boldsymbol{\Omega} = \{H, T\}$

Let $\boldsymbol{X}$ be number of appearance the head, that $P(H) = \frac{1}{2}$ then $P(T) = \frac{1}{2}$

Now let $\boldsymbol{X = 1}$ for appearance the head**,** and let $\boldsymbol{X = 0}$ for without appearance the head**,** e.g. $P(H) = P(X = 1) = \frac{1}{2}$ then

$$P(T) = P(X = 0) = \frac{1}{2} \text{ and } P(X = 1) + P(X = 0) = \mathbf{1}$$

Thus $\boldsymbol{X}$ is discrete random variable since its values are countale values, so that its probability is called probability mass function, th\at is

$$P(X = x) = \begin{cases} \frac{1}{2}, & x = 0,1 \\ 0 & other\ wise \end{cases}$$

Or $P(X = x) = \frac{1}{2} I_{\{0,1\}}(x)$ where

Definition 14 Indicator function Let Ω be any space with points ω and A any subset of Ω. The *indicator function* of A, denoted by $I_A(\cdot)$, is the function with domain Ω and counterdomain equal to the set consisting of the two real numbers 0 and 1 defined by

$$I_A(\omega) = \begin{cases} 1 & \text{if} \quad \omega \in A \\ 0 & \text{if} \quad \omega \notin A. \end{cases}$$

$I_A(\cdot)$ clearly "indicates" the set A. ////

***Howm work*1** What are the properties of the indicator function

EXAMPLE 13 Let the function $f(\cdot)$ be defined by

$$f(x) = \begin{cases} 0 & \text{for} \quad x \leq 0 \\ x & \text{for} \quad 0 < x \leq 1 \\ 2 - x & \text{for} \quad 1 < x \leq 2 \\ 0 & \text{for} \quad 2 < x. \end{cases}$$

By using the indicator function, $f(x)$ can be written as

$$f(x) = xI_{(0,1]}(x) + (2 - x)I_{(1,2]}(x).$$

<u>Boolen Algebra (Algebra)</u>

It is a set of all possible subsets of the sample space Ω,denoted as $\mathcal{A}$, which are called events. Therefore this set has the following properties:

1) $\Omega \in \mathcal{A}$,

2) $A \in \mathcal{A} \rightarrow A^c \in \mathcal{A}$,

3) $A_1, A_2 \in \mathcal{A} \rightarrow A_1 \cup A_2 \in \mathcal{A}$.

<u>Sigma-algebra(σ −algebra)</u>

Let A be non empty set. Then a collection of all subsets of A, denoted $\mathcal{A}, \mathcal{N}, \mathcal{F}$,..

It is called σ **−algebra** iff it has the following properties:

1) $\Omega \in \mathcal{A}$,

2) $A \in \mathcal{A} \rightarrow A^c \in \mathcal{A}$,

3) $A_1, A_2, A_3, \ldots. \in \mathcal{A} \rightarrow A_1 \cup A_2 \cup A_3 \cup \ldots. = \cup_{i=1}^{\infty} A_i \in \mathcal{A}$.

EXAMPLE 7 Toss a penny, nickel, and dime simultaneously, and note which side is up on each. There are eight possible outcomes of this experiment. $\Omega = \{(H, H, H), (H, H, T), (H, T, H), (T, H, H), (H, T, T), (T, H, T), (T, T, H), (T, T, T)\}$. We are using the first position of $(\cdot, \cdot, \cdot)$, called a 3-*tuple*, to record the outcome of the penny, the second position to record the outcome of the nickel, and the third position to record the outcome of the dime. Let $A_i = \{\text{exactly } i \text{ heads}\}$; $i = 0, 1, 2, 3$. For each i, A_i is an event. Note that A_0 and A_3 are each elementary events. Again all subsets of Ω are events; there are $2^8 = 256$ of them. ////

Definition 15 Probability function A *probability function* $P[\cdot]$ is a set function with domain $\mathscr{A}$ (an algebra of events)* and counterdomain the interval $[0, 1]$ which satisfies the following axioms:

(i) $P[A] \geq 0$ for every $A \in \mathscr{A}$.

(ii) $P[\Omega] = 1$.

(iii) If $A_1, A_2, \ldots$ is a sequence of mutually exclusive events in $\mathscr{A}$

(that is, $A_i \cap A_j = \phi$ for $i \neq j$; $i, j = 1, 2, \ldots$) and if $A_1 \cup A_2 \cup \cdots = \bigcup_{i=1}^{\infty} A_i \in \mathscr{A}$, then $P\left[\bigcup_{i=1}^{\infty} A_i\right] = \sum_{i=1}^{\infty} P[A_i]$. ////

where

Definition 10 Disjoint or mutually exclusive Subsets A and B of Ω are defined to be *mutually exclusive* or *disjoint* if $A \cap B = \phi$. Subsets $A_1, A_2, \ldots$ are defined to be *mutually exclusive* if $A_i A_j = \phi$ for every $i \neq j$. ////

EXAMPLE 16 Consider the experiment of tossing two coins, say a penny and a nickel. Let $\Omega = \{(H, H), (H, T), (T, H), (T, T)\}$ where the first component of $(\cdot, \cdot)$ represents the outcome for the penny. Let us model this random experiment by assuming that the four points in Ω are equally likely; that is, assume $P[\{(H, H)\}] = P[\{(H, T)\}] = P[\{(T, H)\}] = P[\{(T, T)\}]$. The following question arises: Is the $P[\cdot]$ function that is implicitly defined by the above really a probability function; that is, does it satisfy the three axioms? It can be shown that it does, and so it is a probability function.

Definition 16 Probability space A *probability space* is the triplet $(\Omega, \mathscr{A}, P[\cdot])$, where Ω is a sample space, $\mathscr{A}$ is a collection (assumed to be an algebra) of events (each a subset of Ω), and $P[\cdot]$ is a probability function with domain $\mathscr{A}$. ////

Finite Sample Space with Equally Points

It is a sample space with finite number of outcomes as N, then it is called

finite sample space with equally points if the probability of each its outcome is $\frac{1}{N}$. Let $\mathbf{P}[.]$ be a probability function, where P: $\boldsymbol{\mathcal{A}} \to [0,1]$ and $\boldsymbol{\mathcal{A}}$ is event space satisfies the following conditions:

1) $\mathbf{P}[\{w_1\}] = \mathbf{P}[\{w_2\}] = \cdots = \mathbf{P}[\{w_N\}]$

2) $A \in \boldsymbol{\mathcal{A}} \to \boldsymbol{P}[A] = \frac{N(A)}{N}$, where $N(A) =$ number of elements that A contains, and $\Omega = \{w_1, w_2, \ldots, w_N\}$. Then it is readily checked that the set function P[·] satisfies the three axioms and hence is a probability function.

Definition 17 Equally likely probability function The probability function $P[\cdot]$ satisfying conditions (i) and (ii) above is defined to be an *equally likely probability function.* ////

EXAMPLE 14 Let Ω be the sample space corresponding to the experiment of tossing two dice, and let $\mathscr{A}$ be the collection of all subsets of Ω. For any $A \in \mathscr{A}$ define $N(A)$ = number of outcomes, or points in Ω, that are in A. Then $N(\phi) = 0$, $N(\Omega) = 36$, and $N(A) = 6$ if A is the event containing those outcomes having a total of seven spots up. ////

Finite sample space without equally likely points We saw for finite sample spaces with equally likely sample points that $P[A] = N(A)/N(\Omega)$ for any event A. For finite sample spaces without equally likely sample points, things are not quite as simple, but we can completely define the values of $P[A]$ for each of the $2^{N(\Omega)}$ events A by specifying the value of $P[\cdot]$ for each of the $N = N(\Omega)$ elementary events. Let $\Omega = \{\omega_1, \ldots, \omega_N\}$, and assume $p_j = P[\{\omega_j\}]$ for $j = 1, \ldots, N$. Since

$$1 = P[\Omega] = P\left[\bigcup_{j=1}^{N}\{\omega_j\}\right] = \sum_{j=1}^{N} P[\{\omega_j\}],$$

$$\sum_{j=1}^{N} p_j = 1.$$

For any event A, define $P[A] = \Sigma p_j$, where the summation is over those ω_j belonging to A. It can be shown that $P[\cdot]$ so defined satisfies the three axioms and hence is a probability function.

EXAMPLE 22 Consider an experiment that has N outcomes, say $\omega_1; \omega_2, \ldots, \omega_N$, where it is known that outcome ω_{j+1} is twice as likely as outcome ω_j, where $j = 1, \ldots, N-1$; that is, $p_{j+1} = 2p_j$, where $p_i = P[\{\omega_i\}]$. Find $P[A_k]$, where $A_k = \{\omega_1, \omega_2, \ldots, \omega_k\}$. Since

$$\sum_{j=1}^{N} p_j = \sum_{j=1}^{N} 2^{j-1} p_1 = p_1(1 + 2 + 2^2 + \cdots + 2^{N-1}) = p_1(2^N - 1) = 1,$$

$$p_1 = \frac{1}{2^N - 1}$$

and

$$p_j = 2^{j-1}/(2^N - 1);$$

hence

$$P[A_k] = \sum_{j=1}^{k} p_j = \sum_{j=1}^{k} 2^{j-1}/(2^N - 1) = \frac{2^k - 1}{2^N - 1}. \qquad ////$$

Conditional Probability and Independence

Definition 18 Conditional probability Let A and B be two events in $\mathscr{A}$ of the given probability space $(\Omega, \mathscr{A}, P[\cdot])$. The *conditional probability* of event A given event B, denoted by $P[A \mid B]$, is defined by

$$P[A \mid B] = \frac{P[AB]}{P[B]} \qquad \text{if} \qquad P[B] > 0, \tag{6}$$

and is left undefined if $P[B] = 0$. ////

Remark A formula that is evident from the definition is $P[AB] = P[A \mid B]P[B] = P[B \mid A]P[A]$ if both $P[A]$ and $P[B]$ are nonzero. This formula relates $P[A \mid B]$ to $P[B \mid A]$ in terms of the unconditional probabilities $P[A]$ and $P[B]$. ////

EXAMPLE 24 Consider the experiment of tossing two coins. Let $\Omega = \{(H, H), (H, T), (T, H), (T, T)\}$, and assume that each point is equally likely. Find (i) the probability of two heads given a head on the first coin and (ii) the probability of two heads given at least one head. Let $A_1 = \{\text{head on first coin}\}$ and $A_2 = \{\text{head on second coin}\}$; then the probability of two heads given a head on the first coin is

$$P[A_1 A_2 \mid A_1] = \frac{P[A_1 A_2 A_1]}{P[A_1]} = \frac{P[A_1 A_2]}{P[A_1]} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}.$$

***Howm work* 2** Does the conditional probability $P[./B]$ satisfy the axioms of the probability f unction?

Theorem 29 Theorem of total probabilities For a given probability space $(\Omega, \mathscr{A}, P[\cdot])$, if $B_1, B_2, \ldots, B_n$ is a collection of mutually disjoint events in $\mathscr{A}$ satisfying $\Omega = \bigcup_{j=1}^{n} B_j$ and $P[B_j] > 0$ for $j = 1, \ldots, n$, then for every $A \in \mathscr{A}$, $P[A] = \sum_{j=1}^{n} P[A \mid B_j]P[B_j]$.

PROOF Note that $A = \bigcup_{j=1}^{n} AB_j$ and the AB_j's are mutually disjoint; hence

$$P[A] = P\left[\bigcup_{j=1}^{n} AB_j\right] = \sum_{j=1}^{n} P[AB_j] = \sum_{j=1}^{n} P[A \mid B_j]P[B_j]. \qquad ////$$

Corollary For a given probability space $(\Omega, \mathscr{A}, P[\cdot])$ let $B \in \mathscr{A}$ satisfy $0 < P[B] < 1$; then for every $A \in \mathscr{A}$

$$P[A] = P[A \mid B]P[B] + P[A \mid \bar{B}]P[\bar{B}]. \qquad ////$$

Theorem 30 Bayes' formula For a given probability space $(\Omega, \mathscr{A}, P[\cdot])$, if $B_1, B_2, \ldots, B_n$ is a collection of mutually disjoint events in $\mathscr{A}$ satisfying $\Omega = \bigcup_{j=1}^{n} B_j$ and $P[B_j] > 0$ for $j = 1, \ldots, n$, then for every $A \in \mathscr{A}$ for which $P[A] > 0$

$$P[B_k \mid A] = \frac{P[A \mid B_k]P[B_k]}{\sum_{j=1}^{n} P[A \mid B_j]P[B_j]}.$$

PROOF

$$P[B_k \mid A] = \frac{P[B_k A]}{P[A]} = \frac{P[A \mid B_k]P[B_k]}{\sum_{j=1}^{n} P[A \mid B_j]P[B_j]}$$

by using both the definition of conditional probability and the theorem of total probabilities. ////

Corollary For a given probability space $(\Omega, \mathscr{A}, P[\cdot])$ let A and $B \in \mathscr{A}$ satisfy $P[A] > 0$ and $0 < P[B] < 1$; then

$$P[B \mid A] = \frac{P[A \mid B]P[B]}{P[A \mid B]P[B] + P[A \mid \bar{B}]P[\bar{B}]}. \qquad ////$$

Theorem 31 Multiplication rule For a given probability space $(\Omega, \mathscr{A}, P[\cdot])$, let $A_1, \ldots, A_n$ be events belonging to $\mathscr{A}$ for which $P[A_1 \cdots A_{n-1}] > 0$; then

$$P[A_1 A_2 \cdots A_n] = P[A_1]P[A_2 \mid A_1]P[A_3 \mid A_1 A_2] \cdots P[A_n \mid A_1 \cdots A_{n-1}].$$

EXAMPLE 27 An urn contains ten balls of which three are black and seven are white. The following game is played: At each trial a ball is selected at random, its color is noted, and it is replaced along with two additional balls of the same color. What is the probability that a black ball is selected in each of the first three trials? Let B_i denote the event that a black ball is selected on the ith trial. We are seeking $P[B_1 B_2 B_3]$. By the multiplication rule,

$$P[B_1 B_2 B_3] = P[B_1]P[B_2 \mid B_1]P[B_3 \mid B_1 B_2] = \tfrac{3}{10} \cdot \tfrac{5}{12} \cdot \tfrac{7}{14} = \tfrac{1}{16}. \qquad ////$$

Definition 19 Independent events For a given probability space $(\Omega, \mathscr{A}, P[\cdot])$, let A and B be two events in $\mathscr{A}$. Events A and B are defined to be *independent* if and only if any one of the following conditions is satisfied:

(i) $P[AB] = P[A]P[B]$.

(ii) $P[A \mid B] = P[A]$ if $P[B] > 0$.

(iii) $P[B \mid A] = P[B]$ if $P[A] > 0$. ////

Definition 20 Independence of several events For a given probability space $(\Omega, \mathscr{A}, P[\cdot])$, let $A_1, A_2, \ldots, A_n$ be n events in $\mathscr{A}$. Events $A_1, A_2, \ldots, A_n$ are defined to be *independent* if and only if

$$P[A_i A_j] = P[A_i]P[A_j] \qquad \text{for } i \neq j$$

$$P[A_i A_j A_k] = P[A_i]P[A_j]P[A_k] \qquad \text{for } i \neq j, j \neq k, i \neq k$$

$$\vdots$$

$$P\left[\bigcap_{i=1}^{n} A_i\right] = \prod_{i=1}^{n} P[A_i]. \qquad ////$$