CHAPTER FIVE

CAPACITORS

**5.1 INTRODUCTION**

So far we have limited our study to resistive circuits. In this chapter, we shall introduce two new and important passive linear circuit elements: the capacitor and the inductor (the inductor is discussed in detail in **Chapter 7**). Unlike resistors, which dissipate energy, capacitors and inductors do not dissipate but store energy, which can be retrieved at a later time. For this reason, capacitors and inductors are called storage elements. We begin by introducing capacitors and describing how to combine them in series or in parallel. Later, we do the same for inductors.

**5.2 CAPACITORS**

A capacitor is a passive element designed to store energy in its electric field. Besides resistors, capacitors are the most common electrical components. Capacitors are used extensively in electronics, communications, computers, and power systems. For example, they are used in the tuning circuits of radio receivers and as dynamic memory elements in computer systems.

A capacitor consists of two conducting plates separated by an insulator (or dielectric).

In many practical applications, the plates may be aluminum foil while the dielectric may be air, ceramic, paper, or mica.

The amount of charge stored, represented by q, is directly proportional to the applied voltage **v** so that

**q = Cv** (5.1)

where C, the constant of proportionality, is known as the capacitance of the capacitor. The unit of capacitance is the farad (**F**), in honor of the English physicist Michael Faraday (1791–1867). From **Eq. (5.1)**, we may derive the following definition.

**Capacitance is the ratio of the charge on one plate of a capacitor to the voltage difference between the two plates, measured in farads (F).**

Note from **Eq. (5.1)** that 1 farad = 1 coulomb/volt.

Although the capacitance **C** of a capacitor is the ratio of the charge **q** per plate to the applied voltage **v**, it does not depend on **q** or **v**. It depends on the physical dimensions of the capacitor. The capacitance is given by

(5.2)

where **A** is the surface area of each plate, **d** is the distance between the plates, and **ε** is the permittivity of the dielectric material between the plates. Typically, capacitors have values in the **picofarad** (**pF**) to **microfarad** (**μF**) range. **Figure 5.1** shows the circuit symbols for fixed and variable capacitors.



**Figure 5.1 Circuit symbols for capacitors: (a) fixed capacitor, (b) variable capacitor.**

To obtain the current-voltage relationship of the capacitor, we take the derivative of both sides of **Eq. (5.1)**. Since

**i =dq/dt** (5.3)

differentiating both sides of **Eq. (5.1)** gives

**i = C dv/dt**  (5.4)

The voltage-current relation of the capacitor can be obtained by integrating both sides of **Eq. (5.4)**. We get

(5.5)

or

(5.6)

where **v(t0) = q(t0)/C** is the voltage across the capacitor at time **t0**.

**Eq. (5.6)** shows that capacitor voltage depends on the past history of the capacitor current. Hence, the capacitor has memory—a property that is often exploited.

The instantaneous power delivered to the capacitor is

(5.7)

The energy stored in the capacitor is therefore

or (5.8)

**Eq. (5.8)** represents the energy stored in the electric field that exists between the plates of the capacitor. This energy can be retrieved, since an ideal capacitor cannot dissipate energy

We should note the following important properties of a capacitor:

1. Note from **Eq. (5.4)** that when the voltage across a capacitor is not changing with time (i.e., dc voltage), the current through the capacitor is zero. Thus,

**A capacitor is an open circuit to dc.**

2. The voltage on the capacitor must be continuous.

**The voltage on a capacitor cannot change abruptly.**

The capacitor resists an abrupt change in the voltage across it.

3. The ideal capacitor does not dissipate energy. It takes power from the circuit when storing energy in its field and returns previously stored energy when delivering power to the circuit.

4. A real, nonideal capacitor has a parallel-model leakage resistance. The leakage resistance may be as high as 100 MΩ and can be neglected for most practical applications.

**Example 5.1:**

(a) Calculate the charge stored on a 3-pF capacitor with 20 V across it.

(b) Find the energy stored in the capacitor.

**Solution:**

(a) Since **q = Cv**, **q = 3 × 10−12 × 20 = 60 pC**

(b) The energy stored is

**Example 5.2:**

The voltage across a 5-μF capacitor is **v(t) = 10 cos 6000t V** Calculate the current through it.

**Solution:**

By definition, the current is

**= −5 × 10−6 × 6000 × 10 sin 6000t = −0.3 sin 6000t A**

**Practice problems:**

1-What is the voltage across a 3-μF capacitor if the charge on one plate is **0.12 mC**? How much energy is stored?

**Answer:** 40 V, 2.4 mJ.

2-If a 10-μF capacitor is connected to a voltage source with **v(t) = 50 sin 2000t V** determine the current through the capacitor.

**Answer:** cos 2000t A.

**5.3 SERIES AND PARALLEL CAPACITORS**

We know from resistive circuits that series-parallel combination is a powerful tool for reducing circuits. This technique can be extended to series-parallel connections of capacitors, which are sometimes encountered. We desire to replace these capacitors by a single equivalent capacitor **Ceq**.

First we obtain the equivalent capacitor **Ceq** of **N** capacitors in parallel,

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**Figure 5.2 (a) Parallel-connected N capacitors, (b) equivalent circuit for the parallel capacitors.**

**Ceq = C1 + C2 + C3 +· · ·+CN** (5.9)

The equivalent capacitance of *N* parallel-connected capacitors is the sum of the individual capacitances.

We observe that capacitors in parallel combine in the same manner as resistors in series.

Now we will obtain **Ceq** of **N** capacitors connected in series



**Figure 5.3 (a) Series-connected N capacitors, (b) equivalent circuit for the series capacitor.**

Where (5.10)

The equivalent capacitance of series-connected capacitors is the reciprocal of the sum of the reciprocals of the individual capacitances.

Note that capacitors in series combine in the same manner as resistors in parallel. For **N = 2** (i.e., two capacitors in series), **Eq. (5.10)** becomes

Or (5.11)

**Example 5.6:**

Find the equivalent capacitance seen between terminals **a** and **b** of the circuit in **Fig. 5.4**.



**Figure 5.4 For Example 6.6.**

**Solution:**

The 20-μF and 5-μF capacitors are in series; their equivalent capacitance is

This 4-μF capacitor is in parallel with the 6-μF and 20-μF capacitors; their combined capacitance is

**4 + 6 + 20 = 30 μF**

This 30-μF capacitor is in series with the 60-μF capacitor. Hence, the equivalent capacitance for the entire circuit is

**Practice problems:**

1- Find the equivalent capacitance seen at the terminals of the circuit in Figure below.

**Answer:** 40 μF.

**5.4 First Order RC Circuit**

Now that we have considered the three passive elements (resistors, capacitors, and inductors, the inductor is discussed in detail in **Chapter 7**), we are prepared to consider circuits that contain various combinations of two or three of the passive elements.

We carry out the analysis of***RC***and***RL*** circuits by applying Kirchhoff’s laws, as we did for resistive circuits. The only difference is that applying Kirchhoff’s laws to purely resistive circuits results in algebraic equations, while applying the laws to ***RC***and ***RL***circuits produces differential equations, which are more difficult to solve than algebraic equations.

The differential equations resulting from analyzing *RC* and *RL* circuits are of the first order. Hence, the circuits are collectively knownas ***first-order***circuits.

A first-order circuit is characterized by a first-order differential equation.

The two types of first-order circuits and thetwo ways of exciting them add up to the four possible situations we willstudy in this chapter.

**5.4 THE SOURCE-FREE *RC* CIRCUIT**

A source-free ***RC***circuit occurs when its dc source is suddenly disconnected. The energy already stored in the capacitor is released to the resistors.

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**Figure 5.5 A source-free *RC* circuit.**

Consider a series combination of a resistor and an initially charged capacitor, as shown in **Fig. 5.5**. Our objective is to determine the circuit response, which, for pedagogic reasons, we assume to be the voltage ***v(t)***across the capacitor. Since the capacitor is initially charged, we can assume that at time ***t* = 0**, the initial voltage is

***v(*0*)* = *V*0** (5. 12)

with the corresponding value of the energy stored as

(5.13)

Applying KCL at the top node of the circuit in **Fig. 5.5**,

***iC* + *iR* = 0** (5.14)

By definition, ***iC* = *C dv/dt***and ***iR* = *v/R***. Thus,

(5.15)

This is a ***first-order differential equation***, since only the first derivative of *v* is involved. After solve it, the capacitor voltage is

(5.16)

This shows that the voltage response of the *RC* circuit is an exponential decay of the initial voltage. Since the response is due to the initial energy stored and the physical characteristics of the circuit and not due to some external voltage or current source, it is called the *natural response* of the circuit.

The natural response of a circuit refers to the behavior (in terms of voltages and currents) of the circuit itself, with no external sources of excitation.

The natural response is illustrated graphically in **Fig. 5.6**. Note that at *t* = 0, we have the correct initial condition as in **Eq. (5.12)**. As *t* increases, the voltage decreases toward zero. The rapidity with which the voltage decreases is expressed in terms of the *time constant*, denoted by the lower case Greek letter tau, ***τ****.*

The time constant of a circuit is the time required for the response to decay by a factor of 1/***e*** or 36.8 percent of its initial value.

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**Figure 5.6 The voltage response of the *RC* circuit.**

This implies that at ***t* = *τ***, **Eq. (5.16)** becomes

or

***τ* = *RC*** (5.17)

In terms of the time constant, **Eq. (5.16)** can be written as

(5.18)

Observe from **Eq. (5.17)** that the smaller the time constant, the more rapidly the voltage decreases, that is, the faster the response. This is illustrated in **Fig. 5.7**. A circuit with a small time constant gives a fast response in that it reaches the steady state (or final state) quickly due to quick dissipation of energy stored, whereas a circuit with a large time constant gives a slow response because it takes longer to reach steady state. At any rate, whether the time constant is small or large, the circuit reaches steady state in five time constants.

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**Figure 5.7 Plot of *v/V*0 = *e*−*t/τ* for various values of the time constant.**

With the voltage *v(t)* in **Eq. (5.18)**, we can find the current *iR(t)*,

(5.19)

The power dissipated in the resistor is

(5.20)

The energy absorbed by the resistor up to time *t* is

(5.21)

Notice the energy that was initially stored in the capacitor is eventually dissipated in the resistor. In summary:

**The Key to Working with a Source - free *RC* Circuit is Finding:**

1. The initial voltage *v(*0*)* = *V*0 across the capacitor.

2. The time constant *τ* .

With these two items, we obtain the response as the capacitor voltage***vC(t)* = *v(t)* =** ***v(*0*)e*−*t/τ***. Once the capacitor voltage is first obtained, other variables (capacitor current ***iC***, resistor voltage ***vR***, and resistor current ***iR***) can be determined. In finding the time constant ***τ* = *RC***, *R* is often the **Thevenin** equivalent resistance at the terminals of the capacitor; that is, we take out the capacitor *C* and find ***R* = *R*Th** at its terminals.

**Example 5.9:** The switch in the circuit in **Fig. 5.8** has been closed for a long time, and it is opened at ***t* = 0**. Find ***v(t)***for ***t* ≥ 0**. Calculate the initial energy stored in the capacitor.

**Solution:**

For ***t <* 0**, the switch is closed; the capacitor is an open circuit to dc, as represented in **Fig. 5.9(a)**. Using voltage division

**Figure 5.8 For Example 5.9.**

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Since the voltage across a capacitor cannot change instantaneously, the voltage across the capacitor at ***t* = 0−** is the same at ***t* = 0**, or

***vC(*0*)* = *V*0 = 15 V**

For *t >* 0, the switch is opened, and we have the ***RC***circuit shown in **Fig. 5.9(b)**. [Notice that the ***RC***circuit in **Fig. 5.9(b)** is source free; the independent source in **Fig. 5.8** is needed to provide ***V*0** or the initial energy in the capacitor.] The **1-*Ω***and **9-Ω**resistors in series give

**Figure 5.9 For Example 5.9:**

**(a) *t <* 0*,* (b) *t >* 0*.***

***R*eq = 1 + 9 = 10 *Ω***

The time constant is

***τ* = *R*eq*C* = 10 × 20 × 10−3 = 0*.*2 s**

Thus, the voltage across the capacitor for ***t* ≥ 0** is

***v(t)* = *vC(*0*)e*−*t/τ* = 15*e*−*t/*0*.*2 V**

or ***v(t)* = 15*e*−5*t* V**

The initial energy stored in the capacitor is

**Practice problems**:

**(1)**Refer to the circuit in Figure below. Let ***vC(*0*)***= 30 V. Determine ***vC***, ***vx***, and ***io***for ***t* ≥ 0**.

**Answer: 30*e*−0*.*25*t* V**, **10*e*−0*.*25*t* V**, **−2*.*5*e*−0*.*25*t***A.

**5.5 STEP RESPONSE OF AN *RC* CIRCUIT**

When the dc source of an ***RC***circuit is suddenly applied, the voltage or current source can be modeled as a step function, and the response is known as a ***step response****.*

The step response of a circuit is its behavior when the excitation is the step function, which may be a voltage or a current source.

The step response is the response of the circuit due to a sudden application of a dc voltage or current source.

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**Figure 5.10 An *RC* circuit with voltage step input.**

Consider the ***RC***circuit in **Fig. 5.10(a)** which can be replaced by the circuit in **Fig. 5.10(b)**, where ***Vs***is a constant, dc voltage source. Again, we select the capacitor voltage as the circuit response to be determined.

We assume an initial voltage ***V*0** on the capacitor, although this is not necessary for the step response. Since the voltage of a capacitor cannot change instantaneously,

***v(*0−*)* = *v(*0+*)* = *V*0** (7.22)

where *v(*0−*)* is the voltage across the capacitor just before switching and *v(*0+*)* is its voltage immediately after switching. Applying **KCL**, we have

(7.23)

where ***v***is the voltage across the capacitor.

Thus,

(5.24)

 This is known as the *complete response* of the *RC* circuit to a sudden application of a dc voltage source, assuming the capacitor is initially charged. The reason for the term “complete” will become evident a little later. Assuming that *Vs > V*0, a plot of ***v(t)***is shown in **Fig. 5.11**.

If we assume that the capacitor is uncharged initially, we set *V*0 = 0 in **Eq. (5.24)** so that

**Figure 5.11 Response of an *RC* circuit with initially charged capacitor.**

(5.25)

Rather than going through the derivations above, there is a systematic approach—or rather, a short-cut method—for finding the step response of an ***RC***or ***RL***circuit. Let us reexamine **Eq. (5.24)**, which is more general than **Eq. (5.25)**. It is evident that ***v(t)***has two components. Thus, we may write

***v* = *vf* + *vn*** (5.26)

We know that ***vn***is the natural response of the circuit, as discussed in **Section 5.2**. Now, ***vf***is known as the *forced* response because it is produced by the circuit when an external “force” is applied (a voltage source in this case).

The natural response or transient response is the circuit’s temporary response that will die out with time.

The forced response or steady-state response is the behavior of the circuit a long time after an external excitation is applied.

The complete response of the circuit is the sum of the natural response and the forced response. Therefore, we may write **Eq. (5.24)** as

***v(t)* = *v(*∞*)* + [*v(*0*)* − *v(*∞*)*]*e*−*t/τ*** (5.27)

where ***v(*0*)***is the initial voltage at ***t* = 0+** and ***v(*∞*)***is the final or steady state value. Thus, to find the step response of an ***RC***circuit requires three things:

1. The initial capacitor voltage ***v (*0*)***.

2. The final capacitor voltage ***v (*∞*)***.

3. The time constant ***τ***.

Note that if the switch changes position at time ***t* = *t*0** instead of at ***t* = 0**, there is a time delay in the response so that **Eq. (5.27)** becomes

***v(t)* = *v(*∞*)* + [*v(t*0*)* − *v(*∞*)*]*e*−*(t*−*t*0*)/τ*** (5.28)

where ***v(t*0*)***is the initial value at ***t* = *t*+0** . Keep in mind that **Eq. (5.27)** or **(5.28)** applies only to step responses, that is, when the input excitation is constant.

**Example 5.10:** The switch in **Fig. 5.12** has been in position *A* for a long time. At *t* = 0, the switch moves to *B*. Determine ***v(t)***for *t >* 0 and calculate its value at *t* = 1 s and 4 s.

**Solution:**

For *t <* 0, the switch is at position *A*. Since ***v***is the same as the voltage across the 5-k*Ω*resistor, the voltage across the capacitor just before *t* = 0 is obtained by voltage division as

**Figure 5.12 For Example 5.10.**

Using the fact that the capacitor voltage cannot change instantaneously,

***v(*0*)* = *v(*0−*)* = *v(*0+*)* = 15 V**

For *t >* 0, the switch is in position *B*. The Thevenin resistance connected to the capacitor is

***R*Th** **= 4 k*Ω***, and the time constant is

***τ* = *R*Th*C* = 4 × 103 × 0*.*5 × 10−3 = 2 s**

Since the capacitor acts like an open circuit to dc at steady state, *v(*∞*)* = 30 V. Thus,

***v(t)* = *v(*∞*)* + [*v(*0*)* − *v(*∞*)*]*e*−*t/τ* = 30 + *(*15 − 30*)e*−*t/*2 = *(*30 − 15*e*−0*.*5*t )* V**

At *t* = 1, ***v(*1*)* = 30 − 15*e*−0*.*5 = 20*.*902 V**

At *t* = 4, ***v(*4*)* = 30 − 15*e*−2 = 27*.*97 V**

Notice that the capacitor voltage is continuous while the resistor current is not.

**PRACTICE PROBLEM:**

(1) Find *v(t)* for*t >* 0 in the circuit in Figure below. Assume the switch has been open for a long time and is closed at *t* = 0. Calculate *v(t)* at *t* = 0*.*5.

**Answer: −5 + 15*e*−2*t* V*,* 0*.*5182 V**.