***Second-Order Differential Equation***

* A **second-order differential equation** is called **linear** if it can be written in the form:

Where the coefficient functions , , , and are continuous functions. Otherwise, it is called**nonlinear**.

* When, for all, such second-order lineardifferential equation is called **homogeneous** and has the following form:

If for some, it is called**nonhomogeneous**.

Two basic facts enable us to solve homogeneous linear equations are stated in the following theorems:

**Theorem 1** If and are both solutions of the linear homogeneous Equation (2) and (and ) are any constants, then the function:

is also a solution of Equation (2).

**Theorem 2** If and are linearly independent solutions of Equation (2)on an interval, and is never 0, then the general solution is given by:

where and are arbitrary constants.

To summarize; the first of these theorems says that if we know two solutions and of such an equation, then the linear combination is a solution.

The other says that the general solution is a linear combination of two linearly independent solutions and. This means that neither nor is a constant multiple of the other. This is very useful because if we know two particular linearly independent solutions, then we know every solution.

***Second-Order Linear Homogeneous Differential Equation with Constant Coefficients***

* If the coefficient functions,, and are constant functions, then Equation (2) become:

Where a, b, and c are constants and a 0.

The exponential function (whereis a constant) has the property that its derivative is a constant multiple of itself. This mean:.Furthermore, .

If we substitute these expressions into Equation (3), we get:

or

So is a solution of if:

Equation (4) is called the **auxiliary equation** or **characteristic equation**.

* Sometimes the roots and of the auxiliary equation can be found by factoring. In other cases they are found by using the quadratic formula:

Therefore, there are three possibilities for cases according to the sign of :

(1)if

The roots and of the auxiliary equation are **real** and **distinct** (Distinct Real Roots).

(2) if

The roots of the auxiliary equation are **real** and **equal** (Repeated Root).

(3) if

The roots and of the auxiliary equation are **complexnumbers** (Complex Roots).

***Case 1: Distinct Real Roots***

If the roots and of the auxiliary equation are real and unequal, then the general solution of is:

Example 1:

Solve the equation.

Solution

The auxiliary equation is:

The roots are: and (Distinct Real Roots).

So, the general solution of the given differential equation is:

Example 2:

Solve the equation.

Solution

The auxiliary equation is:

By using the quadratic formula:

The roots are:and (Distinct Real Roots).

So, the general solution of the given differential equation is:

***Case 2: Repeated Root***

If the roots (repeated root of the auxiliary equation) are real and equal, then the general solution of is:

Example 3:

Solve the equation.

Solution

The auxiliary equation is:

The roots are: (Repeated Root).

So, the general solution of the given differential equation is:

Example 4:

Solve the equation.

Solution

The auxiliary equation is:

The roots are: (Repeated Root).

So, the general solution of the given differential equation is:

***Case 3: Complex Roots***

If the roots of the auxiliary equation are complex numbers, that isandthen the general solution of is:

Example 5:

Solve the equation.

Solution

The auxiliary equation is:

By using the quadratic formula:

The roots are:and (Complex Roots).

So, the general solution of the given differential equation is:

Example 6:

Solve the equation.

Solution

The auxiliary equation is:

The roots are:and (Complex Roots).

So, the general solution of the given differential equation is:

***Initial-Value Problems***

* The general solution of a second-order differential equation always involves two arbitrary constants. In order to determine the value of these constants, it required to specify two initial conditions, {most often and}.

Example 7:

Find the solution of the initial value problem,=0

Solution

The auxiliary equation is:

The roots are: and (Distinct Real Roots).

So, the general solution of the given differential equation is:

By differentiating:

From the given initial conditions =0:

And

Solving for and, we get:

The solution of the initial value problem is then:

Example 8:

Solve the equation

Given that when, and.

Solution

The auxiliary equation is:

The roots are: (Repeated Root).

So, the general solution of the given differential equation is:

By differentiating:

From the given initial conditions, when, and:

And

The solution of the initial value problem is then:

Example 9:

Solve the equation.

Given that when, and.

Solution

The auxiliary equation is:

The roots are:and (Complex Roots).

So, the general solution of the given differential equation is:

By differentiating:

From the given initial conditions, when, and:

And

The solution of the initial value problem is then:

***Second-Order Linear Nonhomogeneous Differential Equation with Constant Coefficient***

* Consider a second-order nonhomogeneous linear differential equation with constant coefficients:

Where a, b, and c are constants and G is a continuous function. The related homogeneous equation

is called the **complementaryequation** and plays an important role in finding the solution of the original nonhomogeneous equation.

**Theorem 3** The general solution of the nonhomogeneous differential Equation (5) can be written as:

Where is a particular solution of Equation (5) and is the general solution of the related homogeneous (complementary) Equation (6).

***The Method of Undetermined Coefficients***

The method of undetermined coefficients is quite simple and it is useful for solving constant coefficient differential equation. All that we need to do is look at and make a guess as to the form of leaving the coefficients undetermined. Place the guess into the differential equation and see if we can determine values of the coefficients. If we can determine values for the coefficients then we guessed correctly, if we can’t find values for the coefficients then we guessed incorrectly.

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| --- | --- |
|  | Initial guess ( |
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**Modification**: If any term of is a solution of the complementary equation, multiply by (or by if necessary).

**Note:**

* If is a product of functions of the previous types:

then:

* If is a sum of functions of the previous types:

then:

Example 10:

Solve the equation.

Solution

The auxiliary equation is:

The roots are: and (Distinct Real Roots).

So, the solution of the complementary equation is:

Since then let particular solution

Sub. in the differential eq. we get:

So,

Therefore,

So, the general solution of the given nonhomogeneous differential equation is:

Example 11:

Solve the equation.

Solution

The auxiliary equation is:

The roots are:and (Complex Roots).

So, the solution of the complementary equation is:

Since then let particular solution

Sub. in the differential eq. we get:

So,

Therefore,

So, the general solution of the given nonhomogeneous differential equation is:

Example 12:

Solve the equation.

Solution

The auxiliary equation is:

The roots are: and (Distinct Real Roots).

So, the solution of the complementary equation is:

Since then let particular solution

Sub. in the differential eq. we get:

So,

Solving for a and b we get:

Therefore,

So, the general solution of the given nonhomogeneous differential equation is:

Example 13:

Solve the equation.

Solution

The auxiliary equation is:

The roots are: and (Distinct Real Roots).

So, the solution of the complementary equation is:

Since

Then let

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Sub. in we get:

+

So,

Therefore,

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Sub. in we get:

So,

Therefore,

So, the general solution of the given nonhomogeneous differential equation is:

Example 14:

Solve the equation.

Solution

The auxiliary equation is:

The roots are:and (Complex Roots).

So, the solution of the complementary equation is:

Since then let particular solution

Sub. in the differential eq. we get:

So,

Therefore,

So, the general solution of the given nonhomogeneous differential equation is:

***The Method of Variation of Parameters***

The method of Variation of Parameters is a general method that can be used in many cases.

Let us consider a second-order nonhomogeneous linear differential equation with constant coefficients:

Moreover, if and are linearly independent solutions of the related homogeneous equation:

In other words, the solution of the complementary equation is:

Then particular solution to the nonhomogeneous differential equation is:

Example 15:

Use the variation of parameters method to solve thefollowing differential equation:

Solution

The auxiliary equation is:

The roots are:and (Complex Roots).

So, the solution of the complementary equation is:

Therefore, we have:

And

Then particular solution to the nonhomogeneous differential equation is:

So, the general solution of the given nonhomogeneous differential equation is:

H.W: Solve the following differential equations using the method of variation of parameters.