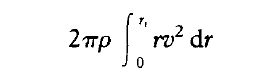
**Laminar flow**

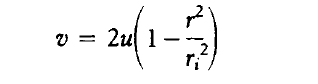
The cases considered so far are ones in which the flow is turbulent and the velocity is nearly uniform over the cross section of the pipe. In laminar flow the curvature of the velocity profile is very pronounced and this must be taken into account in determining the momentum of the fluid.

The momentum flow rate over the cross sectional area of the pipe is easily determined by writing an equation for the momentum flow through an infinitesimal element of area and integrating the equation over the whole cross section. The element of area is an annular strip having inner and outer radii **r** and***,*** the area of which isto the first order in ***.*** The momentum flow rate through this area is isso the momentum flow rate through the whole cross section of the pipe is equal to

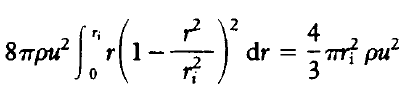


Where ri is the internal radius of the pipe.

It is shown in Example 1.9 that the velocity profile for laminar flow of a Newtonian fluid in a pipe of circular section is parabolic and can be expressed in terms of the volumetric average velocity *u* as:



Therefore the momentum flow rate is equal to



If the velocity had the uniform value *u,* the momentum flow rate would be***.*** Thus for laminar flow of a Newtonian fluid in a pipe the momentum flow rate is greater by a factor of 4/3 than it would be if the same fluid with the same mass flow rate had a uniform velocity. This difference is analogous to the different values of *a* in Bernoulli’s equation (equation 1.14).

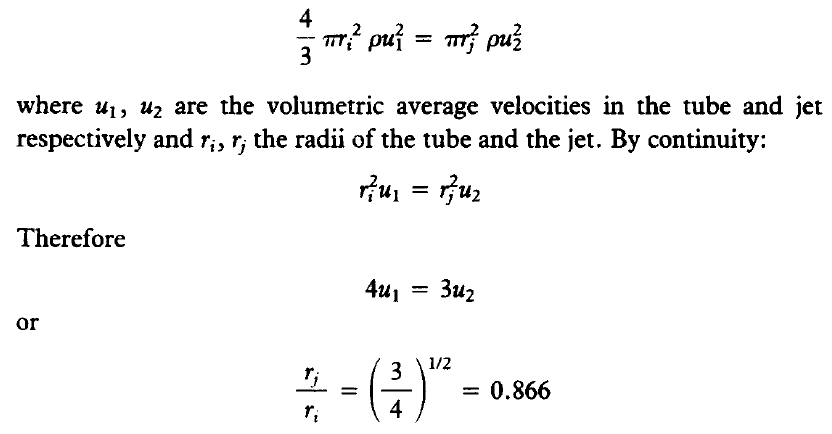
**Example 1.5**

A Newtonian liquid in laminar flow in a horizontal tube emerges into the **air** as a jet from the end of the tube. What is the relationship between the diameters of the jet and the tube?

**Solution**

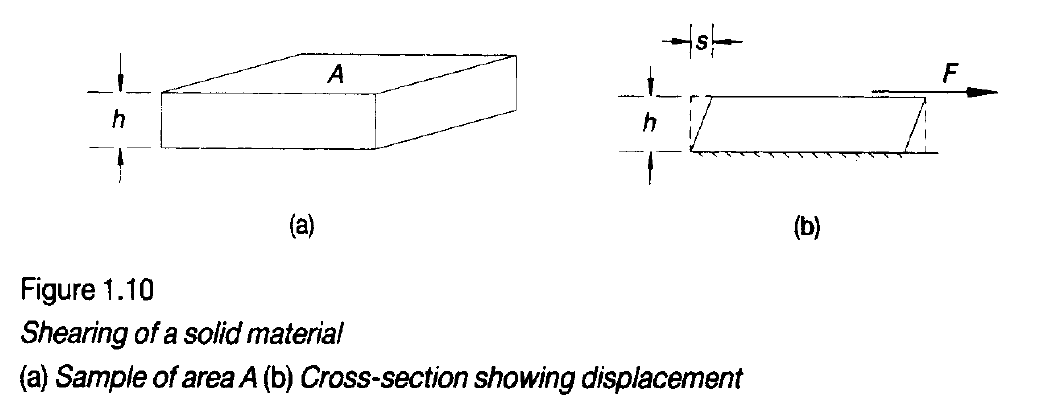
It is assumed that the Reynolds number is sufficiently high for the fluid’s momentum to be dominant and consequently the momentum flow rate in the jet will be the same as that in the tube. On emerging from the tube, there is no wall to maintain the liquid’s parabolic velocity profile and consequently the jet develops a uniform velocity profile.

Equating the momentum of the liquid in the tube to that in the jet gives



Thus, the jet must have a smaller diameter than the tube in order for momentum to be conserved. This result is valid when the liquid’s momentum is dominant. At very low Reynolds numbers, viscous stresses are dominant and the velocity profile starts to change even before the exit plane: in this case the jet diameter is slightly larger than the tube diameter

**Stress in fluids**



The shear force divided by the area over which it acts defines the shear stress



As shown in Figure l.l0 (b), the horizontal displacement of the solid is proportional to the distance from the fixed plate. If the upper plate is displaced a distance s and the solid has a thickness *h* then the shear strain *γ* is defined as



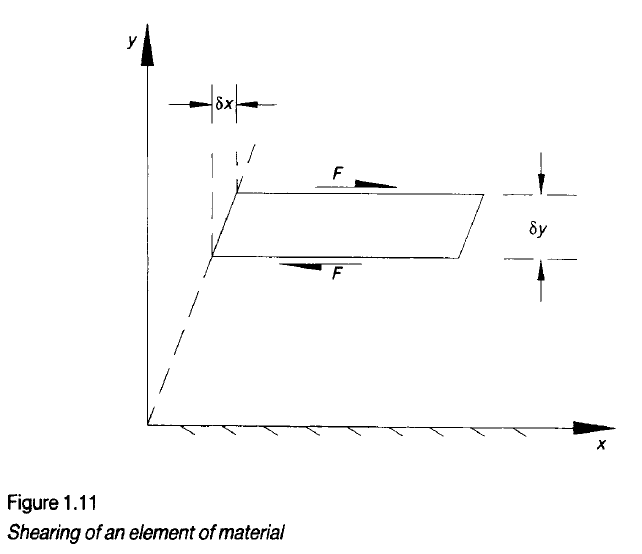
It has been found experimentally that most solid materials exhibit a particularly simple relationship between the shear stress and the shear strain, at least over part of their range of behaviour:



The shear stress is proportional to the shear strain and the constant of proportionality G is known as the shear modulus.

A **thin** slice of the sample, parallel to the plates, is shown in Figure 1.11. The material above the slice is displaced further than the slice **so** the internal force acting on the upper surface of the slice acts in the direction of the applied force. Below the slice, the material is displaced less far and this lower material therefore exerts a force in the opposite direction. If the slice at distancey from the fixed plate is displaced a distance **x** from its unstressed position, the shear strain is equal to **x/y.** It will be seen that the shear strain can also be written as the displacement gradient:





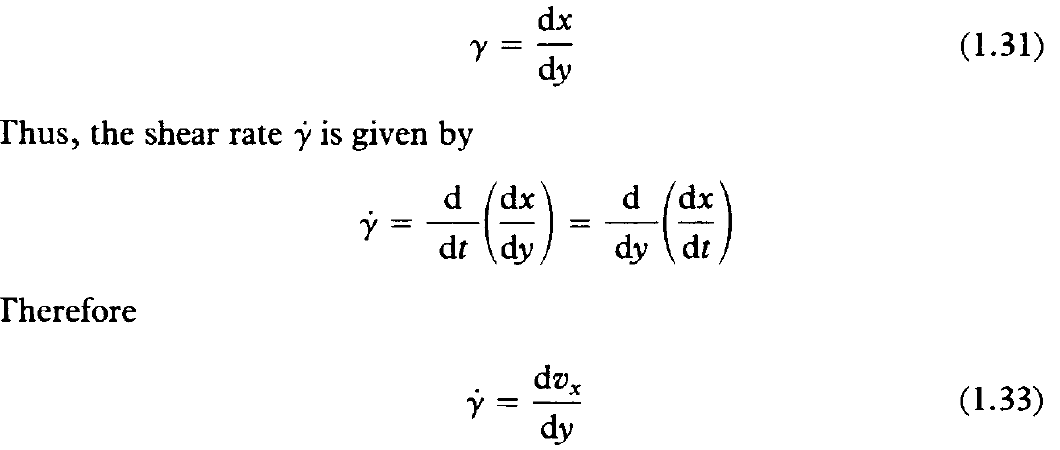
**Newton's law of viscosity**

In contrast to the behaviour of a solid, for a normal fluid the shear stress is independent of the magnitude of the deformation but depends on the rate of change of the deformation. Gases and many liquids exhibit a simple linear relationship between the shear stress **τ** and the rate of shearing:



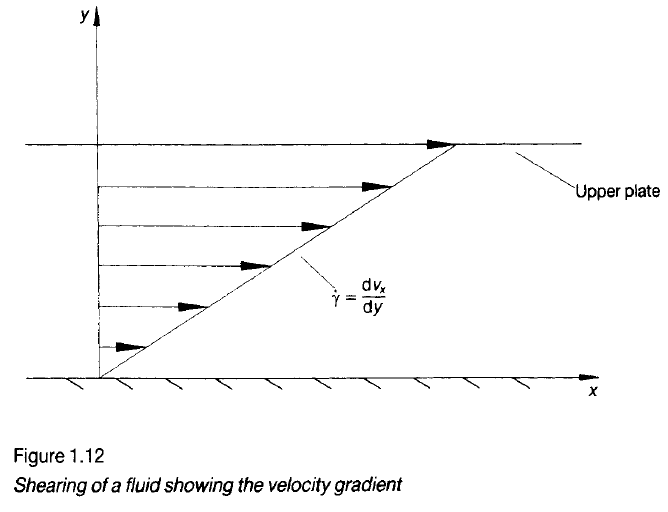
This is a statement of Newton's law of viscosity and the constant of proportionality ***μ*** is known as the coefficient of dynamic viscosity. The rate of change of the shear strain is known as the rate of (shear) strain or the shear rate.

Figure **1.11**, it has been noted that the strain at distance y from the fixed plate can be written as



Equation 1.33 shows that the shear rate at a point is equal to the velocity gradient at that location.

Figure 1.12 shows the flow in terms of the velocity component vx the magnitude of which is indicated by the length of the arrows.

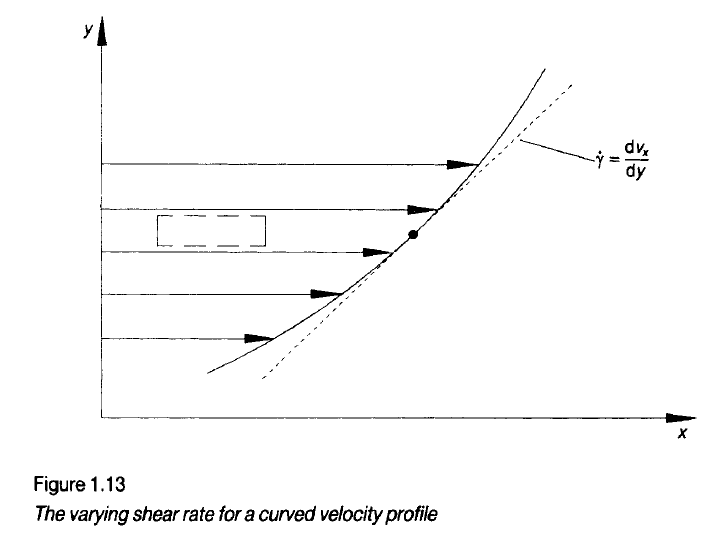


In order to maintain steady flow, the net force acting on the element in the direction of flow must be zero.

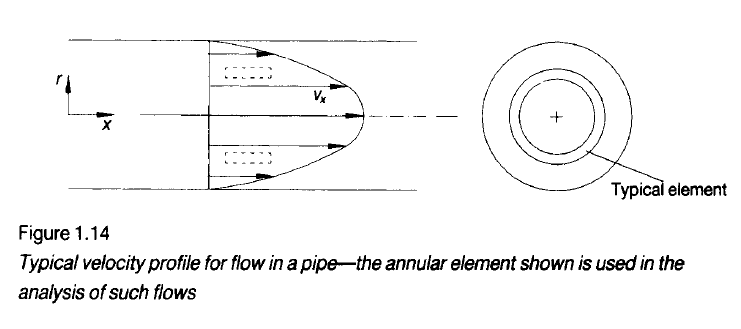
It follows that for this type of flow the shear stress acting on the lower face of the element must have the same magnitude but opposite direction to the shear stress acting on the upper face (see Figure 1.11). Consequently, the magnitude of the shear stress τ is the same at all values of y and from equation 1.32 the shear rate γ. must be constant. In this type of flow, generated by moving the solid boundaries but with no pressure gradient imposed, the velocity profile is linear.

As the velocity profile is curved, the velocity gradient is different at different values of ***y*** and by equation 1.32 the shear stress ***τ*** must vary with y.

shown in Figure 1.13. Clearly, as the velocity profile is curved, the velocity gradient is different at different values of ***y*** and by equation 1.32 the shear stress ***7*** must vary withy. Flows generated by the application of a pressure difference, for‘example over the length of a pipe, have curved velocity profiles.



In the case of flow in a pipe or tube it is natural to use a cylindrical coordinate system as shown in Figure 1.14.

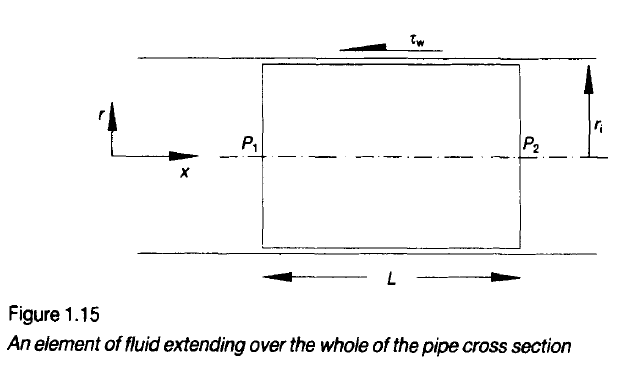


**Example**

Determine the relationship between the shear stress at the wall and the pressure gradient for steady, fully developed, incompressible flow in **a** horizontal pipe.

**Solution**

For the conditions specified, the fluid’s momentum remains constant so the net force acting on the fluid is zero.



Three forces act on the element in the (positive or negative) x-direction:

1. The pressure ***PI*** pushes the fluid in the direction of flow
2. the pressure ***P2*** pushes against the flow
3. The frictional drag between the fluid and the pipe wall acts against the flow

