**Relations Binary relation:**

There are many relations in mathematics :"less than" , "is parallel to ","is a subset of", etc. These relations consider the existence or nonexistence of a certain connection between pairs of objects taken in a definite order. We define a relation simply in terms of ordered pairs of objects.

### Product sets:

Consider two arbitrary sets A and B. The set of all ordered pairs (a,b) where aA and bB is called the product, or cartesian product, of A and B.

A × B = {(a,b) : aA and bB}

**Example**: Let A = {1,2} and B = {a ,b ,c} then

A × B = {(1,a), (1,b),(1,c),(2,a),(2,b),(2,c)}

Also, A × A = {(1, 1), (1, 2), (2, 1), (2, 2)}

- The order in which the sets are considered is important, so A×B ≠ B ×A.

Let A and B be sets. A binary relation, R, from A to B is a subset of A×B. If )x,y) R, we say that x is R-related to y and denote this by xRy

if )x,y) R, we write y and say that x is not R-related to y .

if R is a relation from A to A ,i.e. R is a subset of A × A, then we say that R is a relation on A.

The **domain** of a relation R is the set of all first elements of the ordered pairs which belong to R, and the **range** of R is the set of second elements.

##### Example 1:

Let A = {1, 2, 3, 4}. Define a relation R on A by writing (x, y)  R if x < y. Then R = {(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)}.

##### Example 2:

let A = {1,2,3} and R = {(1,2),(1,3),(3,2)}. Then R is a relation on A since it is a subset of A×A with respect to this relation:

1R2, 1R3, 3R2 but (1,1)R & (2,1)R

The domain of R is {1,3} and The range of R is {2,3} **Example 3**:

Let A = {1, 2, 3}. Define a relation R on A by writing (x, y)  R , such that ab, list the element of R

aRb ↔ ab , a,bA

**** R = {(1,1),(2,1), (2,2), (3,1), (3,2), (3,3)}.

##### Example 4:

A relation on the set Z of integers is “m divides n.” A common notation for this relation is to write m|n when m divides n. Thus 6 | 30 but 25.

##### Representation of relations:

1. By language
2. By ordered pairs
3. By arrow form
4. By matrix form
5. By coordinates
6. By graph form

##### Example:

Let A = {1,2,3}, the relation R on A such that: aRb ↔ a>b; a,bA

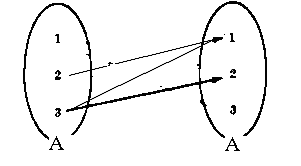
1. By language:

R={(a,b) : a,b A and aRb ↔ a>b}

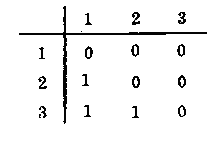
1. By ordered pairs

R = {(2,1),(3,1),(3,2)}

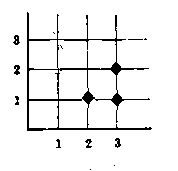
1. By arrow form



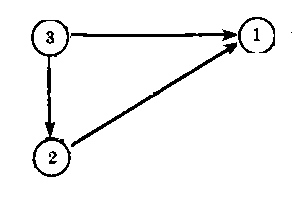
1. By matrix form



1. By coordinates



1. By graph form



### TYPES OF RELATIONS:

**Properties of relations**: Let R be a relation on the set A

1. Reflexive : R is reflexive if :  a A  aRa or (a,a)  R ;  a, b A. . Thus R is not reflexive if there exists a  A such that (a, a)  R.
2. Symmetric : aRb  bRa  a,b A. if whenever *(a, b)* ∈ *R* then *(b, a)* ∈ *R*. Thus *R* is not symmetric if there exists *a, b* ∈ *A* such that *(a, b)* ∈ *R* but *(b, a)*  *R*.
3. Transitive : aRb  bRc  aRc. that is, if whenever *(a, b), (b, c)* ∈ *R*

then *(a, c)* ∈ *R*. Thus *R* is not transitive if there exist *a, b, c* ∈ *R* such that *(a, b)*, *(b,*

*c)* ∈ *R* but *(a, c)*  *R.*

1. Equivalence relation : it is Reflexive & Symmetric & Transitive. That is, *R* is an equivalence relation on *S* if it has the following three properties:

a - For every *a* ∈*S*, *aRa*. b- If *aRb*, then *bRa*.

c- If *aRb* and *bRc*, then *aRc*.

1. Irreflexive :  a A (a,a)  R
2. AntiSymmetric : if aRb and bRa  a=b

the relations , and  are antisymmetric

**Example 5**: Consider the relation of C of set inclusion on any collection of sets:

* 1. A  A for any set, so  is reflexive
  2. A  B dose not imply B  A, so  is not symmetric
  3. If A  B and B  C then A  C, so  is transitive
  4.  is reflexive, not symmetric & transitive, so  is not equivalence relations
  5. A  A, so  is not Irreflexive
  6. If A  B and B  A then A = B, so  is anti-symmetric

**Example 6**: If A ={1,2,3} and R={(1,1),(1,2),(2,1),(2,3)}

Is R equivalence relation ?

1. 2 is in A but (2,2)  R, so R is not reflexive
2. (2,3)  R but (3,2)  R, so R is not symmetric
3. (1,2)  R and (2,3)  R but (1,3)  R, so R is not transitive So R is not Equivalence relation

**Example 7** : What is the properties of the relation =?

1. a=a for any element a  A, so = is reflexive
2. If a = b then b = a, so = is symmetric
3. If a = b and b = c then a = c, so = is transitive

4) = is (reflexive + symmetric + transitive), so = is equivalence

1. a = a, so = is not Irreflexive
2. If a = b and b = a then a = b, so = is anti-symmetric

**Remark:**The properties of being symmetric and being antisymmetric are not negatives of each other. For example, the relation *R* = {(1, 3), (3, 1), (2, 3)} is neither symmetric nor antisymmetric. On the other hand, the relation *R* = {(1, 1), (2, 2)} is both symmetric and antisymmetric.

### -Reflexive Closures

Let *R* be a relation on a set *A*. Then:

*R*  {*(a, a)* | *a*  *A*} is the reflexive closure of *R*. In other words, **reflexive(*R*)** is obtained by simply adding to *R* those elements *(a, a)* in the diagonal which do not already belong to *R*.

### -Symmetric Closures

*R*  *R*−1 is the symmetric closure of *R*. in other words, **symmetric(*R*)** is obtained by adding to *R* all pairs *(b, a)* whenever *(a, b)* belongs to *R*.

**EXAMPLE :** Consider the relation *R* = {*(*1*,* 1*), (*1*,* 3*), (*2*,* 4*), (*3*,* 1*), (*3*,* 3*), (*4*,* 3*)*} on

the set *A* = {1*,* 2*,* 3*,* 4}.Then

reflexive*(R)* = *R*  {*(*2*,* 2*), (*4*,* 4*)*} and

symmetric*(R)* = *R*  {(4, 2), (3, 4)}

### -Transitive Closure

R\* is the transitive closure of *R, where:*

R\*= R  R2  R3  ….  Rn and *R*2 = *R*◦*R* and *Rn* = *Rn*−1◦*R*

**Theorm** : Suppose *A* is a finite set with *n* elements and Let *R* be a relation on a set *A*

with *n* elements. Then : transitive *(R)* = *R*  R2  R3  ….  Rn

**EXAMPLE :** Consider the relation *R* = {*(*1*,* 2*), (*2*,* 3*), (*3*,* 3*)*} on *A* = {1*,* 2*,* 3}. Then:

*R*2 = *R*◦*R* = {*(*1*,* 3*), (*2*,* 3*), (*3*,* 3*)*} and

*R*3 = *R*2◦*R* = {*(*1*,* 3*), (*2*,* 3*), (*3*,* 3*)*} then

transitive *(R)* = {*(*1*,* 2*), (*2*,* 3*), (*3*,* 3*), (*1*,* 3*)*}

##### Inverse relations:

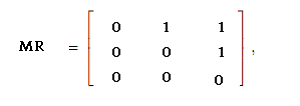
R-1 = {(b,a) : (a,b)  R}

Example 1 :

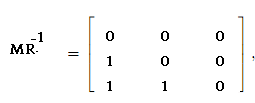
Let R be the following relation on A ={1,2,3} R = {(1,2),(1,3),(2,3)}

**** R-1 = {(2,1),(3,1),(3,2)}

The matrix for R :



and



MR-1 is the transpose of matrix R

**Composition of relations**: Let A, B, C be sets and let :

R : A  B ( R A  B)

S : B  C (S  B C)

There is a relation from A to C denoted by R  S (composition of R and S) : A  C

R  S = {(a,c) :  b  B for which (a,b)  R and (b,c)  S}

Example : let A ={1,2,3,4}

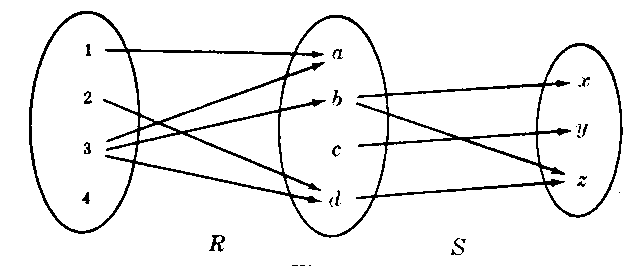
B = {a, b, c, d}

C = {x, y, z}

R = {(1,a),(2,d),(3,a),(3,d),(3,b)}

S = {(b,x),(b,z),(c,y),(d,z)}

Find R  S ? Solution :

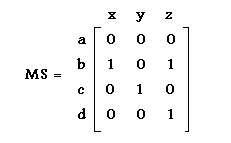
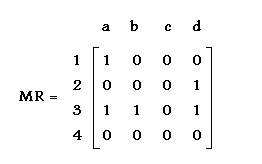
1. The first way by arrow form

There is an arrow (path) from 2 to d which is followed by an arrow from d to z 2Rd and dSz  2(R  S) z

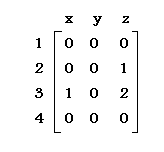
and 3(R◦S)x and 3(R◦S)z

so R  S = {(3,x),(3,z),(2,z)}

1. The second way by matrix:



R  S = MR **.** MS =



R  S = {(2,z),(3,x),(3,z)}

Theorem 2.1: Let A, B, C and D be sets. Suppose R is a relation from A to B, S is a relation from B to C, and

T is a relation from C to D. Then (R ◦ S) ◦ T = R ◦ (S ◦ T )

##### *n*-ARY RELATIONS

All the relations discussed above were binary relations. By an *n*-*ary relation*, we mean a set of ordered *n*-tuples. For any set *S*, a subset of the product set *Sn* is called an *n*-ary relation on *S*. In particular, a subset of S3 is called a *ternary relation* on *S*. **EXAMPLE**

1. Let *L* be a line in the plane. Then “betweenness” is a ternary relation *R* on the points of *L*; that is, *(a, b, c)*  *R,* if *b* lies between *a* and *c* on *L*.
2. The equation *x*2 +*y*2 +*z*2 = 1 determines a ternary relation *T* on the set **R** of real numbers. That is, a triple *(x, y, z)* belongs to *T* if *(x, y, z)* satisfies the equation, which means *(x, y, z)* is the coordinates of a point in **R**3 on the sphere *S* with radius 1 and center at the origin *O* = *(*0*,* 0*,* 0*)*.

##### Home work:

1. Consider the following relations on the set *A* = {1*,* 2*,* 3}:

*R* = {*(*1*,* 1*), (*1*,* 2*), (*1*,* 3*), (*3*,* 3*)*}*,*

*S* = {*(*1*,* 1*)(*1*,* 2*), (*2*,* 1*)(*2*,* 2*), (*3*,* 3*)*}*,*

*T* = {*(*1*,* 1*), (*1*,* 2*), (*2*,* 2*), (*2*,* 3*)*}

 = empty relation

*A*× *A* = universal relation

Determine whether or not each of the above relations on *A* is:

* 1. reflexive; (*b*) symmetric; (*c*) transitive; (*d*) antisymmetric.

1. for the relation *R* = {*(a, a), (a, b), (b, c), (c, c)*} on the set *A* = {*a, b, c*}. Find: (*a*) reflexive(*R*); (*b*) symmetric(*R*); (*c*) transitive(*R*).