مادة شبه الحلقة لمرحلة الماجستير السنة الدراسية 2018-2019 الكورس الاول

المحاضرة الثالثة

Other examples of semirings:

1. If *A* is a nonempty set then the free monoid *A*\* is the set of all finite strings *a*l *a2*. . .*an* of elements of *A* (including the empty string, which is denoted by □). Two strings *a*l *a2*. . .*an* and *b*l *b2* ... *bm* are equal in *A*\* if and only if *n* = *m* and *ai* =*bi* for all 1≤ *i* ≤*m*. We define on *A*\* the operation of concatenation:

*a*l *a2*. . .*an* · *b*l *b2* ... *bk* = *a*l *a2*. . .*an* *b*l *b2* ... *bk*

The set *A\** is a monoid under this operation, the identity element of which is □.

The elements of *A* are often called symbols or letters and the elements of *A\** are called words on these symbols.

Let *A* be a nonempty set. As in Example 1.10(14 in Lecture 2), we can define operations of addition and multiplication on *sub*(*A*\*) as follows:

*L* + *L'* = *L* U *L',* while *LL' =*{*ww'* | *w*  L and *w'*  *L'*}*.*Then (*sub*(*A\**), +, .) is an additively-idempotent entire semiring in which the additive identity is and the multiplicative identity is {□}.

2. If (*M*,+) is a commutative monoid with identity element 0 then the set *End*(*M*)of all endomorphisms of *M* is a semiring under the operations of point wise addition and composition of functions.

We note that *End*0(*M* )= { *End*(*M*)| (0)= 0} is a subsemiring of

*End*(*M*)*.*

3. Let *R*={}. Then (*R*, *min*, *+*)is an additively idempotent commutative semiring in which addition is the operation of taking minimum and multiplication is ordinary addition. As we shall see later, this semiring is important in solving the shortest-path problem in optimization. If *S*=  + {} then (*S*, *min,* +) is a simple subsemiring of (*R*, *min*, +) with infinite element 0.

The semiring *S* has a subsemiring ( {}, *min*,+), known as the tropical semiring, which has important applications in the theory of formal languages and automata theory, including the capture of the nondeterministic complexity of a finite automaton.

Notes:

4. A ring cannot be zerosumfree as a semiring. Indeed, if *R* is a ring then 1 + (-1) = 0 in *R,* while both -1 and 1 are necessarily nonzero.

5. Every additively-idempotent hemiring is zerosumfree. Indeed, if *R* is additively idempotent and if *r* + *r'* = 0 then *r* = *r* + 0 = *r* + (*r* + *r'*)= (*r* + *r*) + *r'* =*r* + *r'* =0 and similarly *r'* = 0.

6. If *R* is a zerosumfree semiring then

*R'* ={0}{*r* *R*| *rb ≠*0 for all 0 *≠b*  *R*}is a subsemiring of *R.*

7.In order for a semiring *R* to be zerosumfree it suffices that there exists one element *t*  *R* satisfying *t* = *t* + 1. Indeed, if such an element exists and if *r* + *r'* =0 then 0 *=* (*r+ r'*)*t* = *rt* + *r't* = *r*(1 *+t*)+ *r'* (1 *+t*)*= r*(1+ *t* )+ *r'* (l+ 1+ *t* )=(*r* + *r'*)+ *r'* + (*r* + *r'*)*t* = *r'*

and so *r* = *r* + *r'* =0 as well.

8. If *R* is simple then, in particular, 1 + 1 = 1 this suffices to show that R is additively idempotent.

9. Conversely, if *R* is additively idempotent then {*a*  *R* | *a* + 1 = 1} is a subsemiring of *R* and so *R* is simple precisely when this subsemiring is all of *R.*

10. If *R* is a simple semiring and if 1 *≠a R* then *ab ≠*1 for all *b*  *R.* Indeed, if *ab* = 1 then *a* =*a*l=*a*(*b*+ 1) =*ab* + *a* = 1 + *a* = 1.

**IDEALS IN SEMIRINGS**

11. Definition. A **left ideal** *I* of a hemiring *R* is a nonempty subset of *R* satisfying the following conditions:

(*1*) *If a, bI then a+ b I*;

(2) If *a* *I* and *r*  *R* then *ra*  *I*;

(3)' *I≠ R.*

12. A **right ideal** of *R* is defined in the analogous manner and an **ideal** of *R* is a subset which is both a left ideal and a right ideal of *R.*

Notes:

13. Ideals are proper, namely *R* is not an ideal of itself.

14. 0 belongs to every [left, right) ideal of *R* and hence {0} is an ideal of *R* contained in every [left, right) ideal of *R.*

15. *U*(*R*)∩*I=*  for every [left, right) ideal of *R.* Any ideal of a semiring *R* is a subhemiring of *R* which is not a subsemiring. Where *U*(*R*) ={*r*  *R*| there exists *r'*  *R* such that *rr*'=1 = *r*'*r*}, the elements of *U*(*R*) are called units.

16. We will denote the set consisting of *R* and all left ideals of *R* by *lideal*(*R*),the set consisting of *R* and all right ideals of *R* by *rideal*(*R*),and the set consisting of *R* and all ideals of *R* by *ideal*(*R*).

17. If *R* is a commutative semiring and if *I= R* \ *U*(*R*)then for *r*  *R* and *a* *I* we surely have *ra*  *I.* Therefore *I* is an ideal of *R* if and only if it is closed under addition. A sufficient condition for this to happen is that if *a*, *bI* then *a* + *b* is either of the form *ra* or *rb* for some *r R.* If *I* is an ideal of *R* then it surely contains every other ideal of *R* and so is the unique maximal ideal of *R.* Inthis case, the commutative semiring *R* is **quasilocal.**

18. The semirings of the form *B*(*n*, *i*) mentioned in Example 1.8 are quasi-local if *i* = 0 and *n* = *ph* for some prime integer *p* and natural number *h,* or if *i* = 1 and *n* - 1=*ph* for some prime integer *p* and natural number *h.*

19. A nonempty subset *A* of a hemiring *R* is **semisubtractive** if and only if *a A∩V(R)* implies that *-a*  *A∩V(R)*;it is **subtractive** if and only if *a*  *A* and *a* + *b*  *A* imply *b*  *A*.

20. Every subtractive subset of *R* surely contains 0.

21. If *R* is a hemiring then the ideal {0}is always subtractive and, as we have noted, is contained in every other subtractive ideal of *R.*

22. Let *R* ={0, 1, *u*}be the idempotent semiring in which 1 + *u* = *u* + 1 = *u.* Then {0, *u*}is an ideal of *R* which is not subtractive.

23. The set 2 of all nonnegative even integers is a subtractive ideal of the semiring of all nonnegative integers.

24. Let *R* be an integral domain with total order compatible with addition and multiplication and let *R+* be the semiring of nonnegative elements of *R.* Then *R+a* is a subtractive ideal of *R+* for all *a*  *R+.*

25. Every ideal of the basic semiring *B*(*n*, *i*) is subtractive if and only if *i* ≤ 1.

26. If *A* is a nonempty subset of a semiring *R* set (0: *A*)= {*r* *R* | *ra* = 0 for all *a A*}*.* If *A≠*{0} then this is a left ideal of *R,* called the left annihilator ideal of *A.* Right annihilator ideals are defined similarly. If *I* is the left annihilator ideal of a nonempty subset *A* of *R* other than {0} then *I* is a subtractive left ideal.

27. Similarly, right annihilator ideals are subtractive right ideals. We note that if *H* is a left ideal of *R* then (0: *H*)is an ideal of *R.* If *aR,* we write (0: *a*)instead of (0: {*a*})*.* Similarly, we note that if *a ≠ b* are elements of *R* then {*r* *R* | *ra* =*rb*}is a left ideal of *R.*

28. **PROPOSITION.** The following conditions on an ideal *I* of a commutative semiring *R* are equivalent:

(*1*) *H* + (0: *I*) = (*HI*: *I*) for all ideals *H* of *R*;

(2) *HI* =*KI* implies that(0: *I*) + *H* = (0: *I*) + *K* for all ideals *H and K* of *R.*

29. **Remark.** In greater generality, if *I* is a left ideal of a semiring *R* and *A* is a nonempty subset of *R,* then (*I*: *A*)= {*r* *R* | *ra*  *I* for each *a*  *A*}is a left ideal of *R* provided that *A* is not a subset of *I*.The right-handed version of this is defined analogously. If *A* = {*a*},we write (*I*: *a*)instead of (*I*:{*a*})*.* It is easily seen that if *I* is a subtractive left ideal of *R* then so is (I: *A*)for any nonempty subset *A* of *R* not a subset of *I*.(If *A* *I* then, of course, (I: *A*)= *R.*)