$$\bar{v}_x = \frac{v_{xi} + v_{xf}}{2} \qquad \text{[for constant } a_x\text{]}$$

Using $\bar{v}_x = \frac{\Delta x}{\Delta t} = \frac{x_f - x_i}{t_f - t_i}$

$$x_f - x_i = \bar{v}_x t \qquad ; \; t_i = 0 \quad \text{and} \quad t_f \equiv t$$

$$\therefore \; x_f - x_i = \frac{1}{2}(v_{xi} + v_{xf})\, t$$

$$\because \; v_{xf} = v_{xi} + a_x t$$

$$\therefore \; x_f - x_i = \frac{1}{2}(v_{xi} + v_{xi} + a_x t)\, t$$

$$\boxed{x_f - x_i = v_{xi} t + \frac{1}{2} a_x t^2}$$

Now use,

$$x_f - x_i = \frac{1}{2}(v_{xi} + v_{xf})\, t$$

$$\because \; t = \frac{v_{xf} - v_{xi}}{a_x}$$

$$x_f - x_i = \frac{1}{2}(v_{xi} + v_{xf})\left(\frac{v_{xf} - v_{xi}}{a_x}\right) = \frac{v_{xf}^2 - v_{xi}^2}{2 a_x}$$

So that,

$$\boxed{v_{xf}^2 = v_{xi}^2 + 2a_x (x_f - x_i)}$$

Summary of equations of motion under constant acceleration:

Equation	information given by equation
$v_{xf} = v_{xi} + a_x t$	velocity as a function of time.
$x_f - x_i = v_{xi} t + \frac{1}{2} a_x t^2$	displacement as a function of velocity and time.
$x_f - x_i = v_{xi} t + \frac{1}{2} a_x t^2$	displacement as a function of time.
$v_{xf}^2 = v_{xi}^2 + 2a_x (x_f - x_i)$	velocity as a function of displacement.

Equations of Motion Derived From Calculus:

From the defining equation for acceleration:

$$a_x = \frac{dv_x}{dt} \Rightarrow dv_x = a_x\, dt$$

By the integral.

$$\int d\nu_x = \int a_x \, dt \quad \Rightarrow \quad \nu_x = \int a_x \, dt + C_1$$

Where C_1 is the constant of the integration.

$$\nu_x = a_x \int dt + C_1 \quad \Rightarrow \quad \nu_x = a_x \, t + C_1$$

If $\nu_x = \nu_{xi}$ at $t = 0$ then,

$\therefore C_1 = \nu_{xi}$

Call $\nu_x = \nu_{xf}$ is the velocity after the time interval t . we obtain:

$$\nu_{xf} = a_x t + \nu_{xi} \quad \Rightarrow \quad \boxed{\nu_{xf} = \nu_{xi} + a_x t}$$

Now let us consider the defining equation for velocity:

$$\nu_x = \frac{dx}{dt} \quad \Rightarrow \quad dx = \nu_x \, dt$$

In integral form:

$$x = \int \nu_x \, dt + C_2$$

$$\because \quad \nu_x = \nu_{xf} = \nu_{xi} + a_x t$$

$$\therefore \quad x = \int (\nu_{xi} + a_x t) \, dt + C_2$$

$$= \int \nu_{xi} \, dt + a_x \int t \, dt + C_2 = \nu_{xi} t + \frac{1}{2} a_x t^2 + C_2$$

We us the initial condition $x = xi$ when $t = 0$, this gives $C2 = xi$.

Now substituting x_f for x , we have:

$$\boxed{x_f = x_i + \nu_{xi} t + \frac{1}{2} a_x t^2}$$

From the defining of the acceleration:

$$a_x = \frac{d\nu_x}{dt} \quad \Rightarrow \quad d\nu_x = a_x \, dt$$

Using the integral with ν_{xf} is the final velocity at $(t = t_f)$ and $(t = t_i = 0)$ the initial velocity ν_{xi} .

$$\therefore \int_{\nu_{xi}}^{\nu_{xf}} d\nu_x = a_x \int_{t_i=0}^{t_f} dt \quad \Rightarrow \quad \nu_{xf} - \nu_{xi} = a_x t$$

So that,

$$\boxed{\nu_{xf} = \nu_{xi} + a_x t}$$

Also, by using the definition of the velocity:

$$\nu_x = \frac{dx}{dt} \quad \Rightarrow \quad dx = \nu_x \, dt = (\nu_{xi} + a_x t) \, dt$$

$$\therefore \int_{x_i}^{x_f} dx = \int \nu_{xi} \, dt + \int a_x \, t \, dt$$

$$\nu_{xi} \int_0^t dt + a_x \int_0^t t \, dt$$

$$\boxed{x_f - x_i = v_{xi}t + \frac{1}{2}a_x t^2}$$

Now,

$$a_x = \frac{dv_x}{dt} = \frac{dv_x}{dt}\frac{dx}{dx} = \frac{dx}{dt}\frac{dv_x}{dx} = v_x \frac{dv_x}{dx}$$

$$\therefore \int_{v_{xi}}^{v_{xf}} v_x\, dv_x = a_x \int_{x_i}^{x_f} dx$$

$$\frac{1}{2}(v_{xf}^2 - v_{xi}^2) = a_x (x_f - x_i)$$

So that,

$$\boxed{v_{xf}^2 = v_{xi}^2 + 2a_x (x_f - x_i)}$$

Freely Falling Bodies:

In the absence of air resistance we find that all bodies, regardless of their size, weight or composition, fall with the same acceleration at the same point of the earth's surface.

The acceleration of a freely falling body is called the acceleration due to gravity and is denoted by the symbol (g). near the earth's surface its magnitude is approximately (9.8 m/Sec^2) and its directed down toward the center of the earth.

Our equations for constant acceleration are applicable here. We simply replace x by y and a_x by g , obtaining:

$$\boxed{\begin{aligned} v_{yf} &= v_{yi} + g t \\ y_f - y_i &= v_{yi}t + \frac{1}{2}g t^2 \\ v_{yf}^2 &= v_{yi}^2 + 2g(y_f - y_i) \end{aligned}}$$

Example (1)

Ideal particle moving in a straight line with position given by: $x = 2.1t^2 + 2.8$ (m)(1) what is a average velocity between t_1=3sec and t_2=5sec (2) instantaneous velocity?

Solution:

(1) t=3sec $x_1=2.1(3)^2+2.8=21.7$m
t=5sec $x_2=2.1(5)^2+2.8=55.3$m

$$\vec{v} = \frac{x_2 - x_1}{t_2 - t_1} = \frac{55.3 - 21.7}{5 - 3} = 16.8\text{m/sec}$$

(2) $v(x) = \frac{dx}{dt} = 4.2t$

Example (2)

A player tosses a baseball up along a y-axis with initial speed of 12m/sec.

(1) How long does the ball take to reach it's maximum?
(2) What is the ball's maximum height above it's release point?
(3) How long does the ball take to reach a point 5m above it's release point?

Solution:

V_o=12m/sec,
V=0

(1) $V= V_o- gt$

$$t = \frac{v - v_0}{-g} = \frac{0 - 12}{-9.8} = 1.2\text{sec}$$

(2)

$$v^2 = v_0^2 - 2g(y - y_0)$$

$$y = \frac{v^2 - v_0^2}{-2g} = \frac{0 - (12)^2}{-2 * 9.8} = 7.3\text{m}$$

(3)

$$y - y_0 = v_0 t - \frac{1}{2} g t^2$$

$$\because y - y_0 = 5m$$

$$\therefore 5 = 12t - \frac{1}{2}(9.8t^2)$$

$$4.9t^2 - 12t + 5 = 0$$

$$t = 0.53 \text{ sec and } t = 1.95 \text{ sec}$$