Chapter 8: Techniques of Integration

8.3 Integration of Rational Functions by Partial Fractions

This section shows how to express a rational function (a quotient of polynomials) as a sum of simpler fractions, called partial fractions, which are easily integrated. For instance, the rational function \( \frac{5x - 3}{x^2 - 2x - 3} \) can be rewritten as

\[
\frac{5x - 3}{x^2 - 2x - 3} = \frac{2}{x + 1} + \frac{3}{x - 3}.
\]
which can be verified algebraically by placing the fractions on the right side over a common denominator \((x + 1)(x - 3)\). The skill acquired in writing rational functions as such a sum is useful in other settings as well (for instance, when using certain transform methods to solve differential equations). To integrate the rational function \((5x - 3)/(x + 1)(x - 3)\) on the left side of our previous expression, we simply sum the integrals of the fractions on the right side:

\[
\int \frac{5x - 3}{(x + 1)(x - 3)} \, dx = \int \frac{2}{x + 1} \, dx + \int \frac{3}{x - 3} \, dx
\]

\[
= 2 \ln |x + 1| + 3 \ln |x - 3| + C.
\]

The method for rewriting rational functions as a sum of simpler fractions is called the method of partial fractions. In the case of the above example, it consists of finding constants \(A\) and \(B\) such that

\[
\frac{5x - 3}{x^2 - 2x - 3} = \frac{A}{x + 1} + \frac{B}{x - 3}.
\]

(Pretend for a moment that we do not know that \(A = 2\) and \(B = 3\) will work.) We call the fractions \(A/(x + 1)\) and \(B/(x - 3)\) partial fractions because their denominators are only part of the original denominator \(x^2 - 2x - 3\). We call \(A\) and \(B\) undetermined coefficients until proper values for them have been found.

To find \(A\) and \(B\), we first clear Equation (1) of fractions, obtaining

\[
5x - 3 = A(x - 3) + B(x + 1) = (A + B)x - 3A + B.
\]

This will be an identity in \(x\) if and only if the coefficients of like powers of \(x\) on the two sides are equal:

\[
A + B = 5, \quad -3A + B = -3.
\]

Solving these equations simultaneously gives \(A = 2\) and \(B = 3\).

**General Description of the Method**

Success in writing a rational function \(f(x)/g(x)\) as a sum of partial fractions depends on two things:

- *The degree of \(f(x)\) must be less than the degree of \(g(x)\).* That is, the fraction must be proper. If it isn’t, divide \(f(x)\) by \(g(x)\) and work with the remainder term. See Example 3 of this section.

- *We must know the factors of \(g(x)\).* In theory, any polynomial with real coefficients can be written as a product of real linear factors and real quadratic factors. In practice, the factors may be hard to find.

Here is how we find the partial fractions of a proper fraction \(f(x)/g(x)\) when the factors of \(g\) are known.
Method of Partial Fractions ($f(x)/g(x)$ Proper)

1. Let $x - r$ be a linear factor of $g(x)$. Suppose that $(x - r)^m$ is the highest power of $x - r$ that divides $g(x)$. Then, to this factor, assign the sum of the $m$ partial fractions:

$$\frac{A_1}{x - r} + \frac{A_2}{(x - r)^2} + \cdots + \frac{A_m}{(x - r)^m}.$$ 

Do this for each distinct linear factor of $g(x)$.

2. Let $x^2 + px + q$ be a quadratic factor of $g(x)$. Suppose that $(x^2 + px + q)^n$ is the highest power of this factor that divides $g(x)$. Then, to this factor, assign the sum of the $n$ partial fractions:

$$\frac{B_1x + C_1}{x^2 + px + q} + \frac{B_2x + C_2}{(x^2 + px + q)^2} + \cdots + \frac{B_nx + C_n}{(x^2 + px + q)^n}.$$ 

Do this for each distinct quadratic factor of $g(x)$ that cannot be factored into linear factors with real coefficients.

3. Set the original fraction $f(x)/g(x)$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of $x$.

4. Equate the coefficients of corresponding powers of $x$ and solve the resulting equations for the undetermined coefficients.

EXAMPLE 1  Distinct Linear Factors

Evaluate

$$\int \frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)} \, dx$$

using partial fractions.

Solution  The partial fraction decomposition has the form

$$\frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)} = \frac{A}{x - 1} + \frac{B}{x + 1} + \frac{C}{x + 3}.$$ 

To find the values of the undetermined coefficients $A$, $B$, and $C$ we clear fractions and get

$$x^2 + 4x + 1 = A(x + 1)(x + 3) + B(x - 1)(x + 3) + C(x - 1)(x + 1)$$

$$= (A + B + C)x^2 + (4A + 2B)x + (3A - 3B - C).$$ 

The polynomials on both sides of the above equation are identical, so we equate coefficients of like powers of $x$ obtaining

Coefficient of $x^2$: $A + B + C = 1$
Coefficient of $x$: $4A + 2B = 4$
Coefficient of $x^0$: $3A - 3B - C = 1$
There are several ways for solving such a system of linear equations for the unknowns \(A\), \(B\), and \(C\), including elimination of variables, or the use of a calculator or computer. Whatever method is used, the solution is \(A = 3/4, B = 1/2, \) and \(C = -1/4\). Hence we have

\[
\int \frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)} \, dx = \int \left[ \frac{3}{4x - 1} + \frac{1}{2x + 1} - \frac{1}{4x + 3} \right] \, dx
\]

\[
= \frac{3}{4} \ln |x - 1| + \frac{1}{2} \ln |x + 1| - \frac{1}{4} \ln |x + 3| + K,
\]

where \(K\) is the arbitrary constant of integration (to avoid confusion with the undetermined coefficient we labeled as \(C\)).

**EXAMPLE 2**  A Repeated Linear Factor

Evaluate \(\int \frac{6x + 7}{(x + 2)^2} \, dx\).

**Solution**  First we express the integrand as a sum of partial fractions with undetermined coefficients.

\[
\frac{6x + 7}{(x + 2)^2} = \frac{A}{x + 2} + \frac{B}{(x + 2)^2}
\]

\[
6x + 7 = A(x + 2) + B
\]

Equating coefficients of corresponding powers of \(x\) gives

\[A = 6 \quad \text{and} \quad 2A + B = 12 + B = 7, \quad \text{or} \quad A = 6 \quad \text{and} \quad B = -5.\]

Therefore,

\[
\int \frac{6x + 7}{(x + 2)^2} \, dx = \int \left( \frac{6}{x + 2} - \frac{5}{(x + 2)^2} \right) \, dx
\]

\[
= 6 \int \frac{dx}{x + 2} - 5 \int (x + 2)^{-2} \, dx
\]

\[
= 6 \ln |x + 2| + 5(x + 2)^{-1} + C
\]

**EXAMPLE 3**  Integrating an Improper Fraction

Evaluate \(\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} \, dx\).

**Solution**  First we divide the denominator into the numerator to get a polynomial plus a proper fraction.

\[
x^2 - 2x - 3 \left( \frac{2x}{2x^3 - 4x^2 - x - 3} \right) = \frac{2x}{2x^3 - 4x^2 - 6x}{5x - 3}
\]
Then we write the improper fraction as a polynomial plus a proper fraction.

\[
\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = 2x + \frac{5x - 3}{x^2 - 2x - 3}
\]

We found the partial fraction decomposition of the fraction on the right in the opening example, so

\[
\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} \, dx = \int 2x \, dx + \int \frac{5x - 3}{x^2 - 2x - 3} \, dx
\]

\[
= \int 2x \, dx + \int \frac{2}{x + 1} \, dx + \int \frac{3}{x - 3} \, dx
\]

\[
= x^2 + 2 \ln |x + 1| + 3 \ln |x - 3| + C.
\]

A quadratic polynomial is **irreducible** if it cannot be written as the product of two linear factors with real coefficients.

**EXAMPLE 4** Integrating with an Irreducible Quadratic Factor in the Denominator

Evaluate

\[
\int \frac{-2x + 4}{(x^2 + 1)(x - 1)^2} \, dx
\]

using partial fractions.

**Solution** The denominator has an irreducible quadratic factor as well as a repeated linear factor, so we write

\[
\frac{-2x + 4}{(x^2 + 1)(x - 1)^2} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x - 1} + \frac{D}{(x - 1)^2}.
\]

Clearing the equation of fractions gives

\[
-2x + 4 = (Ax + B)(x - 1)^2 + C(x - 1)(x^2 + 1) + D(x^2 + 1)
\]

\[
= (A + C)x^3 + (-2A + B - C + D)x^2
\]

\[
+ (A - 2B + C)x + (B - C + D).
\]

Equating coefficients of like terms gives

| Coefficients of \(x^3\): | \(0 = A + C\) |
| Coefficients of \(x^2\): | \(0 = -2A + B - C + D\) |
| Coefficients of \(x^1\): | \(-2 = A - 2B + C\) |
| Coefficients of \(x^0\): | \(4 = B - C + D\) |

We solve these equations simultaneously to find the values of \(A, B, C,\) and \(D\):

\[-4 = -2A, \quad A = 2\]  Subtract fourth equation from second.

\[C = -A = -2\]  From the first equation

\[B = 1\]  \(A = 2\) and \(C = -2\) in third equation.

\[D = 4 - B + C = 1.\]  From the fourth equation

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We substitute these values into Equation (2), obtaining
\[
\frac{-2x + 4}{(x^2 + 1)(x - 1)^2} = \frac{2x + 1}{x^2 + 1} - \frac{2}{x - 1} + \frac{1}{(x - 1)^2}.
\]
Finally, using the expansion above we can integrate:
\[
\int \frac{-2x + 4}{(x^2 + 1)(x - 1)^2} \, dx = \int \left( \frac{2x + 1}{x^2 + 1} - \frac{2}{x - 1} + \frac{1}{(x - 1)^2} \right) \, dx
\]
\[
= \int \left( \frac{2x}{x^2 + 1} + \frac{1}{x^2 + 1} - \frac{2}{x - 1} + \frac{1}{(x - 1)^2} \right) \, dx
\]
\[
= \ln (x^2 + 1) + \tan^{-1} x - 2 \ln |x - 1| - \frac{1}{x - 1} + C. \quad \blacksquare
\]

**EXAMPLE 5** A Repeated Irreducible Quadratic Factor

Evaluate
\[
\int \frac{dx}{x(x^2 + 1)^2}.
\]

**Solution** The form of the partial fraction decomposition is
\[
\frac{1}{x(x^2 + 1)^2} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1} + \frac{Dx + E}{(x^2 + 1)^2}
\]
Multiplying by \(x(x^2 + 1)^2\), we have
\[
1 = A(x^2 + 1)^2 + (Bx + C)x(x^2 + 1) + (Dx + E)x
\]
\[
= A(x^4 + 2x^2 + 1) + B(x^4 + x^2) + C(x^3 + x) + Dx^2 + Ex
\]
\[
= (A + B)x^4 + Cx^3 + (2A + B + D)x^2 + (C + E)x + A
\]
If we equate coefficients, we get the system
\[
A + B = 0, \quad C = 0, \quad 2A + B + D = 0, \quad C + E = 0, \quad A = 1.
\]
Solving this system gives \(A = 1, \quad B = -1, \quad C = 0, \quad D = -1, \quad E = 0\). Thus,
\[
\int \frac{dx}{x(x^2 + 1)^2} = \int \left[ \frac{1}{x} + \frac{-x}{x^2 + 1} + \frac{-x}{(x^2 + 1)^2} \right] \, dx
\]
\[
= \int \frac{dx}{x} - \int \frac{x \, dx}{x^2 + 1} - \int \frac{x \, dx}{(x^2 + 1)^2}
\]
\[
= \int \frac{dx}{x} - \frac{1}{2} \int \frac{du}{u} - \frac{1}{2} \int \frac{du}{u^2}
\]
\[
= \ln |x| - \frac{1}{2} \ln |u| + \frac{1}{2u} + K
\]
\[
= \ln |x| - \frac{1}{2} \ln (x^2 + 1) + \frac{1}{2(x^2 + 1)} + K
\]
\[
= \ln \frac{|x|}{\sqrt{x^2 + 1}} + \frac{1}{2(x^2 + 1)} + K. \quad \blacksquare
\]
The Heaviside "Cover-up" Method for Linear Factors

When the degree of the polynomial $f(x)$ is less than the degree of $g(x)$ and $g(x) = (x - r_1)(x - r_2) \cdots (x - r_n)$ is a product of $n$ distinct linear factors, each raised to the first power, there is a quick way to expand $f(x)/g(x)$ by partial fractions.

**EXAMPLE 6** Using the Heaviside Method

Find $A$, $B$, and $C$ in the partial-fraction expansion

$$
\frac{x^2 + 1}{(x - 1)(x - 2)(x - 3)} = \frac{A}{x - 1} + \frac{B}{x - 2} + \frac{C}{x - 3}.
$$

**Solution** If we multiply both sides of Equation (3) by $(x - 1)$ to get

$$
\frac{x^2 + 1}{(x - 2)(x - 3)} = A + \frac{B(x - 1)}{x - 2} + \frac{C(x - 1)}{x - 3}
$$

and set $x = 1$, the resulting equation gives the value of $A$:

$$
\frac{(1)^2 + 1}{(1 - 2)(1 - 3)} = A + 0 + 0,
$$

$$
A = 1.
$$

Thus, the value of $A$ is the number we would have obtained if we had covered the factor $(x - 1)$ in the denominator of the original fraction

$$
\frac{x^2 + 1}{(x - 1)(x - 2)(x - 3)}
$$

and evaluated the rest at $x = 1$:

$$
A = \frac{(1)^2 + 1}{(x - 1)} \frac{1}{(1 - 2)(1 - 3)} = \frac{2}{(-1)(-2)} = 1.
$$

Similarly, we find the value of $B$ in Equation (3) by covering the factor $(x - 2)$ in Equation (4) and evaluating the rest at $x = 2$:

$$
B = \frac{(2)^2 + 1}{(2 - 1)} \frac{1}{(x - 2)(2 - 3)} = \frac{5}{(1)(-1)} = -5.
$$

Finally, $C$ is found by covering the $(x - 3)$ in Equation (4) and evaluating the rest at $x = 3$:

$$
C = \frac{(3)^2 + 1}{(3 - 1)(3 - 2)} \frac{1}{(x - 3)} = \frac{10}{(2)(1)} = 5.
$$

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Heaviside Method

1. Write the quotient with $g(x)$ factored:

$$\frac{f(x)}{g(x)} = \frac{f(x)}{(x - r_1)(x - r_2) \cdots (x - r_n)}.$$ 

2. Cover the factors $(x - r_i)$ of $g(x)$ one at a time, each time replacing all the uncovered $x$’s by the number $r_i$. This gives a number $A_i$ for each root $r_i$:

$$A_1 = \frac{f(r_1)}{(r_1 - r_2) \cdots (r_1 - r_n)}$$

$$A_2 = \frac{f(r_2)}{(r_2 - r_1)(r_2 - r_3) \cdots (r_2 - r_n)}$$

$$\vdots$$

$$A_n = \frac{f(r_n)}{(r_n - r_1)(r_n - r_2) \cdots (r_n - r_{n-1})}.$$ 

3. Write the partial-fraction expansion of $f(x)/g(x)$ as

$$\frac{f(x)}{g(x)} = \frac{A_1}{(x - r_1)} + \frac{A_2}{(x - r_2)} + \cdots + \frac{A_n}{(x - r_n)}.$$

**Example 7** Integrating with the Heaviside Method

Evaluate

$$\int \frac{x + 4}{x^3 + 3x^2 - 10x} \, dx.$$ 

**Solution** The degree of $f(x) = x + 4$ is less than the degree of $g(x) = x^3 + 3x^2 - 10x$, and, with $g(x)$ factored,

$$\frac{x + 4}{x^3 + 3x^2 - 10x} = \frac{x + 4}{x(x - 2)(x + 5)}.$$ 

The roots of $g(x)$ are $r_1 = 0$, $r_2 = 2$, and $r_3 = -5$. We find

$$A_1 = \frac{0 + 4}{(0 - 2)(0 + 5)} = \frac{4}{(-2)(5)} = -\frac{2}{5}$$

Cover

$$A_2 = \frac{2 + 4}{2(x - 2)(2 + 5)} = \frac{6}{(2)(7)} = \frac{3}{7}$$

Cover

$$A_3 = \frac{-5 + 4}{(-5)(-5 - 2)(x + 5)} = \frac{-1}{(-5)(-7)} = -\frac{1}{35}.$$ 

Cover
Therefore,
\[
\frac{x + 4}{x(x - 2)(x + 5)} = -\frac{2}{5x} + \frac{3}{7(x - 2)} - \frac{1}{35(x + 5)},
\]
and
\[
\int \frac{x + 4}{x(x - 2)(x + 5)} \, dx = -\frac{2}{5} \ln |x| + \frac{3}{7} \ln |x - 2| - \frac{1}{35} \ln |x + 5| + C.
\]

**Other Ways to Determine the Coefficients**

Another way to determine the constants that appear in partial fractions is to differentiate, as in the next example. Still another is to assign selected numerical values to \( x \).

**EXAMPLE 8** Using Differentiation

Find \( A \), \( B \), and \( C \) in the equation
\[
\frac{x - 1}{(x + 1)^3} = \frac{A}{x + 1} + \frac{B}{(x + 1)^2} + \frac{C}{(x + 1)^3}.
\]

**Solution** We first clear fractions:
\[
x - 1 = A(x + 1)^2 + B(x + 1) + C.
\]
Substituting \( x = -1 \) shows \( C = -2 \). We then differentiate both sides with respect to \( x \), obtaining
\[
1 = 2A(x + 1) + B.
\]
Substituting \( x = -1 \) shows \( B = 1 \). We differentiate again to get \( 0 = 2A \), which shows \( A = 0 \). Hence,
\[
\frac{x - 1}{(x + 1)^3} = \frac{1}{(x + 1)^2} - \frac{2}{(x + 1)^3}.
\]
In some problems, assigning small values to \( x \) such as \( x = 0, \pm 1, \pm 2 \), to get equations in \( A, B, \) and \( C \) provides a fast alternative to other methods.

**EXAMPLE 9** Assigning Numerical Values to \( x \)

Find \( A \), \( B \), and \( C \) in
\[
\frac{x^2 + 1}{(x - 1)(x - 2)(x - 3)} = \frac{A}{x - 1} + \frac{B}{x - 2} + \frac{C}{x - 3}.
\]

**Solution** Clear fractions to get
\[
x^2 + 1 = A(x - 2)(x - 3) + B(x - 1)(x - 3) + C(x - 1)(x - 2).
\]
Then let $x = 1, 2, 3$ successively to find $A, B, \text{ and } C$:

$x = 1$: $\quad (1)^2 + 1 = A(-1)(-2) + B(0) + C(0)$

$\quad 2 = 2A$

$\quad A = 1$

$x = 2$: $\quad (2)^2 + 1 = A(0) + B(1)(-1) + C(0)$

$\quad 5 = -B$

$\quad B = -5$

$x = 3$: $\quad (3)^2 + 1 = A(0) + B(0) + C(2)(1)$

$\quad 10 = 2C$

$\quad C = 5.$

Conclusion:

$$\frac{x^2 + 1}{(x - 1)(x - 2)(x - 3)} = \frac{1}{x - 1} - \frac{5}{x - 2} + \frac{5}{x - 3}. \quad \blacksquare$$
### Excerpts from Thomas' Calculus

**EXERCISES 8.3**

**Expanding Quotients into Partial Fractions**

Expand the quotients in Exercises 1–8 by partial fractions.

<table>
<thead>
<tr>
<th>Exercise</th>
<th>Quotient</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>(\frac{5x-13}{(x-3)(x-2)})</td>
</tr>
<tr>
<td>2.</td>
<td>(\frac{5x-7}{x^2-3x+2})</td>
</tr>
<tr>
<td>3.</td>
<td>(\frac{x+4}{(x+1)^2})</td>
</tr>
<tr>
<td>4.</td>
<td>(\frac{2x+2}{x^2-2x+1})</td>
</tr>
<tr>
<td>5.</td>
<td>(\frac{z+1}{z^2(z-1)})</td>
</tr>
<tr>
<td>6.</td>
<td>(\frac{z}{z^3-z^2-6z})</td>
</tr>
<tr>
<td>7.</td>
<td>(\frac{t^2+8}{t^3-5t+6})</td>
</tr>
<tr>
<td>8.</td>
<td>(\frac{r^4+9}{t^4+9t^2})</td>
</tr>
</tbody>
</table>

**Nonrepeated Linear Factors**

In Exercises 9–16, express the integrands as a sum of partial fractions and evaluate the integrals.

<table>
<thead>
<tr>
<th>Exercise</th>
<th>Integrand</th>
</tr>
</thead>
<tbody>
<tr>
<td>9.</td>
<td>(\int \frac{dx}{1-x^2})</td>
</tr>
<tr>
<td>10.</td>
<td>(\int \frac{dx}{x^2+2x})</td>
</tr>
<tr>
<td>11.</td>
<td>(\int \frac{x+4}{x^2+5x-6} , dx)</td>
</tr>
<tr>
<td>12.</td>
<td>(\int \frac{2x+1}{x^2-7x+12} , dx)</td>
</tr>
<tr>
<td>13.</td>
<td>(\int_4^8 \frac{y , dy}{y^2-2y-3})</td>
</tr>
<tr>
<td>14.</td>
<td>(\int_{\sqrt{2}}^{1} \frac{y+4}{y^2+y} , dy)</td>
</tr>
<tr>
<td>15.</td>
<td>(\int dt)</td>
</tr>
<tr>
<td>16.</td>
<td>(\int \frac{x+3}{2x^3-8x} , dx)</td>
</tr>
</tbody>
</table>

**Repeated Linear Factors**

In Exercises 17–20, express the integrands as a sum of partial fractions and evaluate the integrals.

<table>
<thead>
<tr>
<th>Exercise</th>
<th>Integrand</th>
</tr>
</thead>
<tbody>
<tr>
<td>17.</td>
<td>(\int_0^1 \frac{x^3 , dx}{x^2+2x+1})</td>
</tr>
<tr>
<td>18.</td>
<td>(\int_{-1}^0 \frac{x^3 , dx}{x^2-2x+1})</td>
</tr>
</tbody>
</table>

**Irreducible Quadratic Factors**

In Exercises 21–28, express the integrands as a sum of partial fractions and evaluate the integrals.

<table>
<thead>
<tr>
<th>Exercise</th>
<th>Integrand</th>
</tr>
</thead>
<tbody>
<tr>
<td>19.</td>
<td>(\int \frac{dx}{(x^2-1)^2})</td>
</tr>
<tr>
<td>20.</td>
<td>(\int \frac{x^2 , dx}{(x-1)(x^2+2x+1)})</td>
</tr>
</tbody>
</table>

**Improper Fractions**

In Exercises 29–34, perform long division on the integrand, write the proper fraction as a sum of partial fractions, and then evaluate the integral.

<table>
<thead>
<tr>
<th>Exercise</th>
<th>Integrand</th>
</tr>
</thead>
<tbody>
<tr>
<td>21.</td>
<td>(\int_0^1 \frac{dx}{(x+1)(x^2+1)})</td>
</tr>
<tr>
<td>22.</td>
<td>(\int \frac{3t^2+t+4}{t^3+t} , dt)</td>
</tr>
<tr>
<td>23.</td>
<td>(\int \frac{y^2+2y+1}{(y^2+1)^2} , dy)</td>
</tr>
<tr>
<td>24.</td>
<td>(\int \frac{8x^2+8x+2}{(4x^2+1)^2} , dx)</td>
</tr>
<tr>
<td>25.</td>
<td>(\int \frac{2x+2}{(s^2+1)(s-1)^3} , ds)</td>
</tr>
<tr>
<td>26.</td>
<td>(\int \frac{s^4+81}{s^2+9} , ds)</td>
</tr>
<tr>
<td>27.</td>
<td>(\int \frac{2\theta^3+5\theta^2+8\theta+4}{(\theta^2+2\theta+2)^2} , d\theta)</td>
</tr>
<tr>
<td>28.</td>
<td>(\int \frac{\theta^4-4\theta^3+2\theta^2-3\theta+1}{(\theta^2+1)^3} , d\theta)</td>
</tr>
</tbody>
</table>

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Chapter 8: Techniques of Integration

Evaluating Integrals
Evaluate the integrals in Exercises 35–40.

35. \[ \int e^t dt \]
36. \[ \int \frac{e^{2t} + 2e^t - e^t}{e^{2t} + 1} dt \]
37. \[ \int \frac{\cos y dy}{\sin^2 y + \sin y - 6} \]
38. \[ \int \frac{\sin \theta d\theta}{\cos^2 \theta + \cos \theta - 2} \]
39. \[ \int \frac{(x - 2)^2 \tan^{-1} (2x) - 12x^3 - 3x}{(4x^2 + 1)(x - 2)^2} dx \]
40. \[ \int \frac{(x + 1)^3 \tan^{-1} (3x) + 9x^3 + x}{(9x^2 + 1)(x + 1)^2} dx \]

Initial Value Problems
Solve the initial value problems in Exercises 41–44 for \( x \) as a function of \( t \).

41. \( (t^2 - 3t + 2) \frac{dx}{dt} = 1 \quad (t > 2), \quad x(3) = 0 \)
42. \( (3t^4 + 4t^2 + 1) \frac{dx}{dt} = 2\sqrt{3}, \quad x(1) = -\pi\sqrt{3}/4 \)
43. \( (t^2 + 2t) \frac{dx}{dt} = 2x + 2 \quad (t, x > 0), \quad x(1) = 1 \)
44. \( (t + 1) \frac{dx}{dt} = x^2 + 1 \quad (t > -1), \quad x(0) = \pi/4 \)

Applications and Examples
In Exercises 45 and 46, find the volume of the solid generated by revolving the shaded region about the indicated axis.

45. The \( x \)-axis

46. The \( y \)-axis

47. Find, to two decimal places, the \( x \)-coordinate of the centroid of the region in the first quadrant bounded by the \( x \)-axis, the curve \( y = \tan^{-1} x \), and the line \( x = \sqrt{3} \).

48. Find the \( x \)-coordinate of the centroid of this region to two decimal places.

49. Social diffusion
Sociologists sometimes use the phrase “social diffusion” to describe the way information spreads through a population. The information might be a rumor, a cultural fad, or news about a technical innovation. In a sufficiently large population, the number of people \( x \) who have the information is treated as a differentiable function of time \( t \), and the rate of diffusion, \( dx/dt \), is assumed to be proportional to the number of people who have the information times the number of people who do not. This leads to the equation

\[ \frac{dx}{dt} = kx(N - x), \]

where \( N \) is the number of people in the population.

Suppose \( t \) is in days, \( k = 1/250 \), and two people start a rumor at time \( t = 0 \) in a population of \( N = 1000 \) people.

a. Find \( x \) as a function of \( t \).

b. When will half the population have heard the rumor? (This is when the rumor will be spreading the fastest.)

50. Second-order chemical reactions
Many chemical reactions are the result of the interaction of two molecules that undergo a change to produce a new product. The rate of the reaction typically depends on the concentrations of the two kinds of molecules. If \( a \) is the amount of substance \( A \) and \( b \) is the amount of substance \( B \) at time \( t = 0 \), and if \( x \) is the amount of product at time \( t \), then the rate of formation of \( x \) may be given by the differential equation

\[ \frac{dx}{dt} = k(a - x)(b - x), \]

or

\[ \frac{1}{(a - x)(b - x)} \frac{dx}{dt} = k, \]

where \( k \) is a constant for the reaction. Integrate both sides of this equation to obtain a relation between \( x \) and \( t \) (a) if \( a = b \), and (b) if \( a \neq b \). Assume in each case that \( x = 0 \) when \( t = 0 \).

51. An integral connecting \( \pi \) to the approximation 22/7

a. Evaluate \( \int_0^1 \frac{x^4(x - 1)^4}{x^2 + 1} dx \).

b. How good is the approximation \( \pi \approx 22/7 \)? Find out by expressing \( \left( \frac{22}{7} - \pi \right) \) as a percentage of \( \pi \).
e. Graph the function \( y = \frac{x^4(x - 1)^4}{x^2 + 1} \) for \( 0 \leq x \leq 1 \). Experiment with the range on the \( y \)-axis set between 0 and 1, then between 0 and 0.5, and then decreasing the range until the graph can be seen. What do you conclude about the area under the curve?

52. Find the second-degree polynomial \( P(x) \) such that \( P(0) = 1 \), \( P'(0) = 0 \), and

\[ \int \frac{P(x)}{x^3(x - 1)^2} \, dx \]

is a rational function.