Rotation About the *y*-Axis :

In this case, the circular cross-section is :

$$A(y) = \pi[\text{radius}]^2 = \pi[R(y)]^2$$

EX :

Find the volume of the solid generated by revolving the region between the y-axis and the curve $x = 2/y,\ 1 \le y \le 4$ **about the y – axis .**

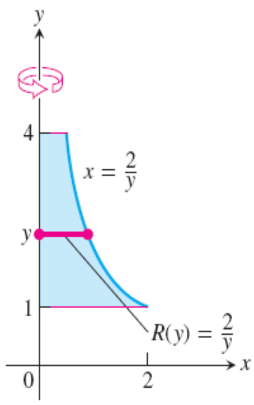
Sol: We draw figures showing the region, a typical radius, and the generated solid .

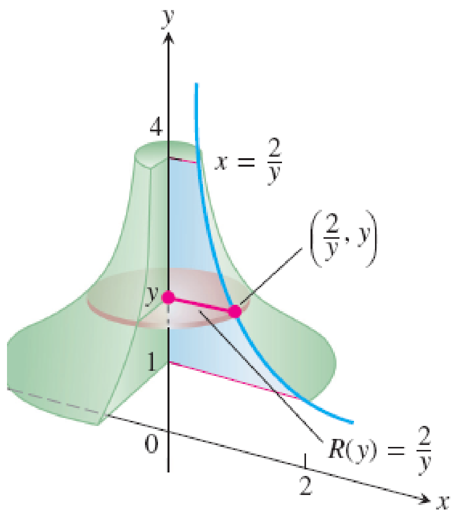
$$V = \int_1^4 \pi[R(y)]^2\, dy$$

$$= \int_1^4 \pi\left(\frac{2}{y}\right)^2 dy$$

$$= \pi\int_1^4 \frac{4}{y^2}\, dy = 4\pi\left[-\frac{1}{y}\right]_1^4 = 4\pi\left[\frac{3}{4}\right]$$

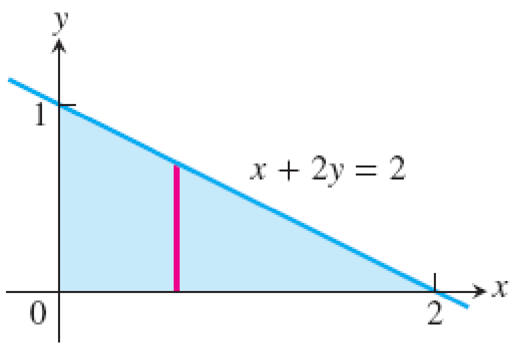
$$= 3\pi .$$



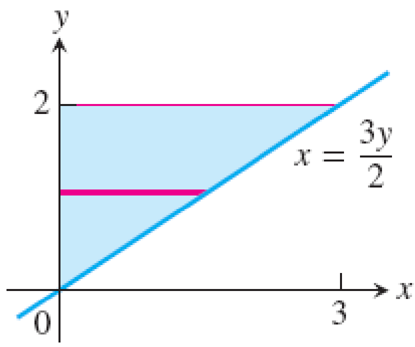


EX : find the volume of the solid generated by revolving the shaded region about the given axis.

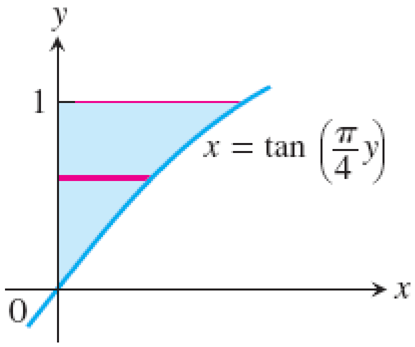
About the *x*-axis



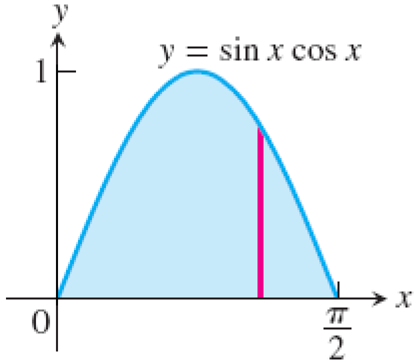
About the *y*-axis



About the y-axis



About the x-axis



SOL :

$$R(x) = y = 1 - \tfrac{x}{2} \;\Rightarrow\; V = \int_0^2 \pi[R(x)]^2\,dx = \pi\int_0^2 \left(1 - \tfrac{x}{2}\right)^2 dx = \pi\int_0^2 \left(1 - x + \tfrac{x^2}{4}\right) dx = \pi\left[x - \tfrac{x^2}{2} + \tfrac{x^3}{12}\right]_0^2$$
$$= \pi\left(2 - \tfrac{4}{2} + \tfrac{8}{12}\right) = \tfrac{2\pi}{3}$$

$$R(y) = x = \tfrac{3y}{2} \;\Rightarrow\; V = \int_0^2 \pi[R(y)]^2\,dy = \pi\int_0^2 \left(\tfrac{3y}{2}\right)^2 dy = \pi\int_0^2 \tfrac{9}{4}y^2\,dy = \pi\left[\tfrac{3}{4}y^3\right]_0^2 = \pi\cdot\tfrac{3}{4}\cdot 8 = 6\pi$$

$$R(x) = \tan\left(\tfrac{\pi}{4}y\right);\; u = \tfrac{\pi}{4}y \;\Rightarrow\; du = \tfrac{\pi}{4}\,dy \;\Rightarrow\; 4\,du = \pi\,dy;\; y = 0 \;\Rightarrow\; u = 0,\; y = 1 \;\Rightarrow\; u = \tfrac{\pi}{4};$$
$$V = \int_0^1 \pi[R(y)]^2\,dy = \pi\int_0^1 \left[\tan\left(\tfrac{\pi}{4}y\right)\right]^2 dy = 4\int_0^{\pi/4} \tan^2 u\,du = 4\int_0^{\pi/4}(-1 + \sec^2 u)\,du = 4[-u + \tan u]_0^{\pi/4}$$

$$= 4\left(-\tfrac{\pi}{4} + 1 - 0\right) = 4 - \pi$$

$$R(x) = \sin x\cos x;\; R(x) = 0 \;\Rightarrow\; a = 0 \text{ and } b = \tfrac{\pi}{2} \text{ are the limits of integration; } V = \int_0^{\pi/2} \pi[R(x)]^2\,dx$$
$$= \pi\int_0^{\pi/2} (\sin x\cos x)^2\,dx = \pi\int_0^{\pi/2} \tfrac{(\sin 2x)^2}{4}\,dx;\; \left[u = 2x \;\Rightarrow\; du = 2\,dx \;\Rightarrow\; \tfrac{du}{8} = \tfrac{dx}{4};\; x = 0 \;\Rightarrow\; u = 0,\right.$$
$$\left. x = \tfrac{\pi}{2} \;\Rightarrow\; u = \pi\right] \;\rightarrow\; V = \pi\int_0^{\pi} \tfrac{1}{8}\sin^2 u\,du = \tfrac{\pi}{8}\left[\tfrac{u}{2} - \tfrac{1}{4}\sin 2u\right]_0^{\pi} = \tfrac{\pi}{8}\left[\left(\tfrac{\pi}{2} - 0\right) - 0\right] = \tfrac{\pi^2}{16}$$

EX : Find the volumes of the solids generated by revolving the regions bounded by the lines and curves about the y-axis.

The region enclosed by $x = \sqrt{5}y^2,\quad x = 0,\quad y = -1,\quad y = 1$

The region enclosed by $x = y^{3/2},\quad x = 0,\quad y = 2$

The region enclosed by $x = \sqrt{2\sin 2y},\quad 0 \le y \le \pi/2,\quad x = 0$

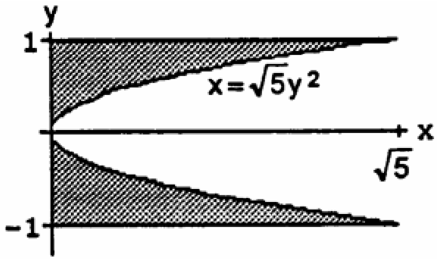
The region enclosed by $x = \sqrt{\cos(\pi y/4)},\quad -2 \le y \le 0,$
$x = 0$

$x = 2/(y + 1),\quad x = 0,\quad y = 0,\quad y = 3$

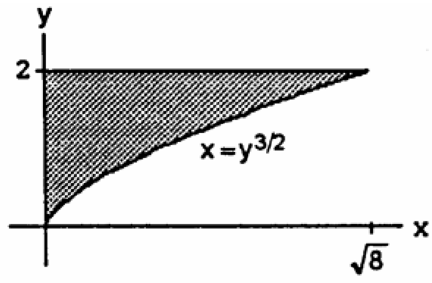
$x = \sqrt{2y}/(y^2 + 1),\quad x = 0,\quad y = 1$

SOL :

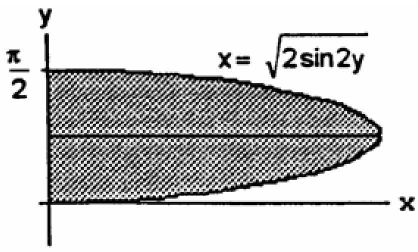
$$R(y)=\sqrt{5}\cdot y^2 \;\Rightarrow\; V=\int_{-1}^{1}\pi[R(y)]^2\,dy=\pi\int_{-1}^{1}5y^4\,dy$$
$$=\pi\left[y^5\right]_{-1}^{1}=\pi[1-(-1)]=2\pi$$



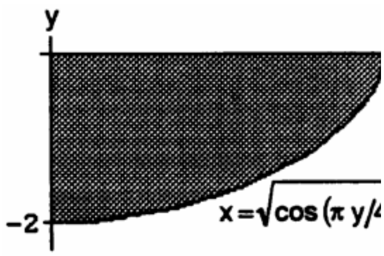
$$R(y)=y^{3/2} \;\Rightarrow\; V=\int_0^2\pi[R(y)]^2\,dy=\pi\int_0^2 y^3\,dy$$
$$=\pi\left[\frac{y^4}{4}\right]_0^2=4\pi$$



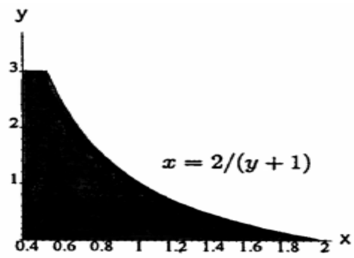
$$R(y)=\sqrt{2\sin 2y} \;\Rightarrow\; V=\int_0^{\pi/2}\pi[R(y)]^2\,dy$$
$$=\pi\int_0^{\pi/2}2\sin 2y\,dy=\pi\left[-\cos 2y\right]_0^{\pi/2}$$
$$=\pi[1-(-1)]=2\pi$$



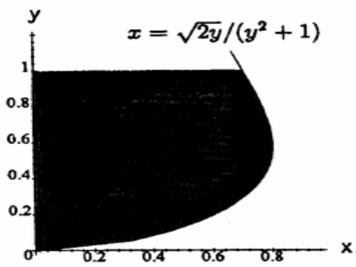
$$R(y)=\sqrt{\cos\frac{\pi y}{4}} \;\Rightarrow\; V=\int_{-2}^{0}\pi[R(y)]^2\,dy$$
$$=\pi\int_{-2}^{0}\cos\left(\frac{\pi y}{4}\right)dy\;=4\left[\sin\frac{\pi y}{4}\right]_{-2}^{0}=4[0-(-1)]=4$$



$$R(y)=\frac{2}{y+1} \;\Rightarrow\; V=\int_0^3\pi[R(y)]^2\,dy=4\pi\int_0^3\frac{1}{(y+1)^2}\,dy$$
$$=4\pi\left[\frac{-1}{y+1}\right]_0^3=4\pi\left[-\frac{1}{4}-(-1)\right]=3\pi$$



$$R(y)=\frac{\sqrt{2y}}{y^2+1} \;\Rightarrow\; V=\int_0^1\pi[R(y)]^2\,dy=\pi\int_0^1 2y\,(y^2+1)^{-2}\,dy;$$
$$[u=y^2+1 \;\Rightarrow\; du=2y\,dy;\; y=0 \;\Rightarrow\; u=1,\; y=1 \;\Rightarrow\; u=2]$$
$$\rightarrow\; V=\pi\int_1^2u^{-2}\,du=\pi\left[-\frac{1}{u}\right]_1^2=\pi\left[-\frac{1}{2}-(-1)\right]=\frac{\pi}{2}$$



EX : Find the volume of the solid generated by revolving each region about the y-axis:

The region enclosed by the triangle with vertices (1, 0), (2, 1), and (1, 1)

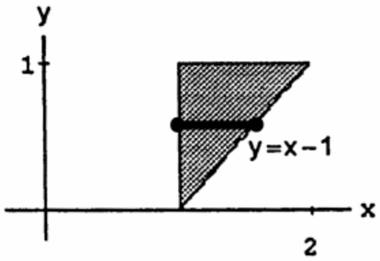
The region enclosed by the triangle with vertices (0, 1), (1, 0), and (1, 1)

The region in the first quadrant bounded above by the parabola $y = x^2$, below by the x-axis, and on the right by the line $x = 2$

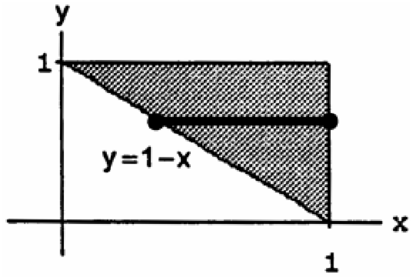
The region in the first quadrant bounded on the left by the circle $x^2 + y^2 = 3$, on the right by the line $x = \sqrt{3}$, and above by the line $y = \sqrt{3}$

$r(y) = 1$ and $R(y) = 1 + y$

$$\Rightarrow V = \int_0^1 \pi\left([R(y)]^2 - [r(y)]^2\right) dy$$
$$= \pi \int_0^1 [(1+y)^2 - 1]\, dy = \pi \int_0^1 (1 + 2y + y^2 - 1)\, dy$$
$$= \pi \int_0^1 (2y + y^2)\, dy = \pi \left[y^2 + \frac{y^3}{3}\right]_0^1 = \pi\left(1 + \frac{1}{3}\right) = \frac{4\pi}{3}$$

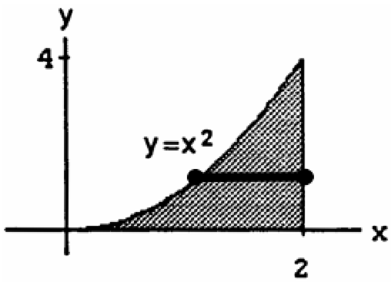


$$R(y) = 1 \text{ and } r(y) = 1 - y \Rightarrow V = \int_0^1 \pi\left([R(y)]^2 - [r(y)]^2\right) dy$$
$$= \pi \int_0^1 [1 - (1-y)^2]\, dy = \pi \int_0^1 [1 - (1 - 2y + y^2)]\, dy$$
$$= \pi \int_0^1 (2y - y^2)\, dy = \pi \left[y^2 - \frac{y^3}{3}\right]_0^1 = \pi\left(1 - \frac{1}{3}\right) = \frac{2\pi}{3}$$



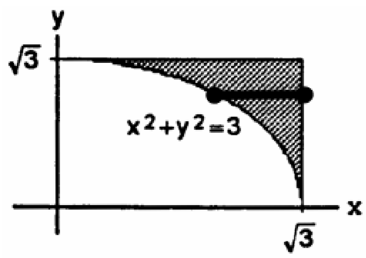
$R(y) = 2$ and $r(y) = \sqrt{y}$

$$\Rightarrow V = \int_0^4 \pi\left([R(y)]^2 - [r(y)]^2\right) dy$$
$$= \pi \int_0^4 (4 - y)\, dy = \pi \left[4y - \frac{y^2}{2}\right]_0^4 = \pi(16 - 8) = 8\pi$$

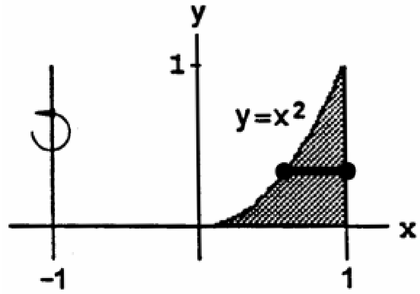


$R(y) = \sqrt{3}$ and $r(y) = \sqrt{3 - y^2}$

$$\Rightarrow V = \int_0^{\sqrt{3}} \pi\left([R(y)]^2 - [r(y)]^2\right) dy$$
$$= \pi \int_0^{\sqrt{3}} [3 - (3 - y^2)]\, dy = \pi \int_0^{\sqrt{3}} y^2\, dy$$
$$= \pi \left[\frac{y^3}{3}\right]_0^{\sqrt{3}} = \pi\sqrt{3}$$



$R(y) = 2$ and $r(y) = 1 + \sqrt{y}$



$\Rightarrow V = \int_0^1 \pi\left([R(y)]^2 - [r(y)]^2\right) dy$

$= \pi \int_0^1 \left[4 - \left(1 + \sqrt{y}\right)^2\right] dy$

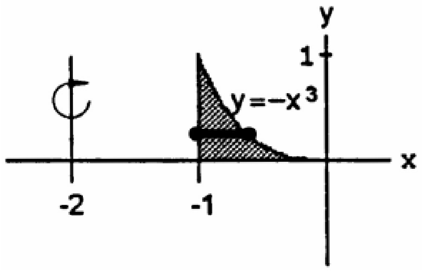
$= \pi \int_0^1 \left(4 - 1 - 2\sqrt{y} - y\right) dy$

$= \pi \int_0^1 \left(3 - 2\sqrt{y} - y\right) dy$

$= \pi \left[3y - \frac{4}{3} y^{3/2} - \frac{y^2}{2}\right]_0^1$

$= \pi \left(3 - \frac{4}{3} - \frac{1}{2}\right) = \pi \left(\frac{18-8-3}{6}\right) = \frac{7\pi}{6}$

$R(y) = 2 - y^{1/3}$ and $r(y) = 1$



$\Rightarrow V = \int_0^1 \pi\left([R(y)]^2 - [r(y)]^2\right) dy$

$= \pi \int_0^1 \left[\left(2 - y^{1/3}\right)^2 - 1\right] dy$

$= \pi \int_0^1 \left(4 - 4y^{1/3} + y^{2/3} - 1\right) dy$

$= \pi \int_0^1 \left(3 - 4y^{1/3} + y^{2/3}\right) dy$

$= \pi \left[3y - 3y^{4/3} + \frac{3y^{5/3}}{5}\right]_0^1 = \pi \left(3 - 3 + \frac{3}{5}\right) = \frac{3\pi}{5}$

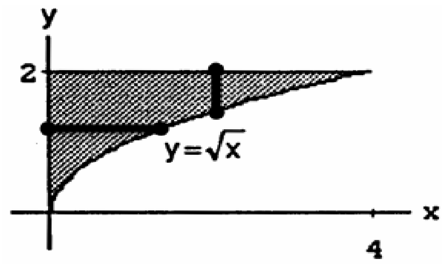
EX :

Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 2$ and $x = 0$ about

a. the x-axis. **b.** the y-axis.

c. the line $y = 2$. **d.** the line $x = 4$.

SOL :



(a) $r(x) = \sqrt{x}$ and $R(x) = 2$

$\Rightarrow V = \int_0^4 \pi\left([R(x)]^2 - [r(x)]^2\right) dx$

$= \pi \int_0^4 (4 - x)\, dx = \pi \left[4x - \frac{x^2}{2}\right]_0^4 = \pi(16 - 8) = 8\pi$

(b) $r(y) = 0$ and $R(y) = y^2$

$\Rightarrow V = \int_0^2 \pi\left([R(y)]^2 - [r(y)]^2\right) dy$

$= \pi \int_0^2 y^4\, dy = \pi \left[\frac{y^5}{5}\right]_0^2 = \frac{32\pi}{5}$

(c) $r(x) = 0$ and $R(x) = 2 - \sqrt{x} \Rightarrow V = \int_0^4 \pi\left([R(x)]^2 - [r(x)]^2\right) dx = \pi \int_0^4 \left(2 - \sqrt{x}\right)^2 dx$

$= \pi \int_0^4 \left(4 - 4\sqrt{x} + x\right) dx = \pi \left[4x - \frac{8x^{3/2}}{3} + \frac{x^2}{2}\right]_0^4 = \pi \left(16 - \frac{64}{3} + \frac{16}{2}\right) = \frac{8\pi}{3}$

(d) $r(y) = 4 - y^2$ and $R(y) = 4 \Rightarrow V = \int_0^2 \pi\left([R(y)]^2 - [r(y)]^2\right) dy = \pi \int_0^2 \left[16 - \left(4 - y^2\right)^2\right] dy$

$= \pi \int_0^2 \left(16 - 16 + 8y^2 - y^4\right) dy = \pi \int_0^2 \left(8y^2 - y^4\right) dy = \pi \left[\frac{8}{3} y^3 - \frac{y^5}{5}\right]_0^2 = \pi \left(\frac{64}{3} - \frac{32}{5}\right) = \frac{224\pi}{15}$