VOLUMES

DEFINITION : Volume

The **volume** of a solid of known integrable cross-sectional area $A(x)$ from x = a to x = b is the integral of A from a to b :

$$V = \int_a^b A(x)\,dx$$

Calculating the Volume of a Solid :

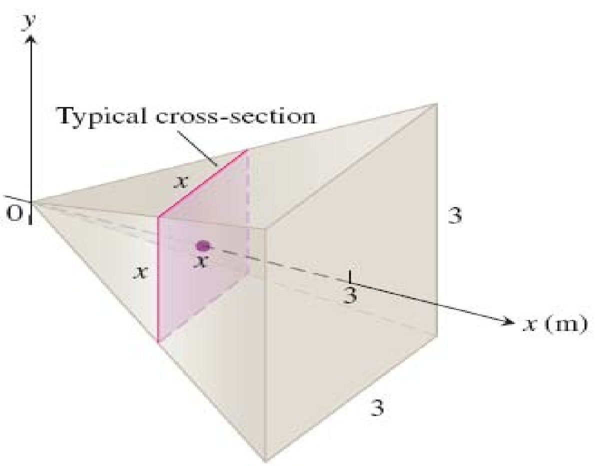
1. Sketch the solid .
2. Find a formula for A(x), the area .
3. Find the limits of integration.
4. Integrate A(x) .

EXAMPLE 1 : Volume of a Pyramid

A pyramid 3 m high has a square base that is 3 m on a side. The cross-section of the pyramid perpendicular to the altitude x down from the vertex is a square x on a side. Find the volume of the pyramid.

Solution :

1. A sketch. We draw the pyramid with its altitude along the x-axis and its vertex at the origin .



2. A formula for A(x). The cross-section at x is a square x meters on a side, so its area is

$$A(x) = x^2$$

3. The limits of integration. The squares lie on the planes from x = 0 to x = 3

4. Integrate to find the volume.

$$V = \int_0^3 A(x)\,dx = \int_0^3 x^2\,dx = \frac{x^3}{3}\Big]_0^3 = 9 \text{ m}^3$$

Find the volume of Solids (by using disc method) :

To find the volume of a solid like the one shown in the figures of the following examples ,we need only observe that the cross-sectional area A(x) is the area of a disk of radius R(x), the distance of the planar region’s boundary from the axis . Then the area is :

$$A(x) = \pi(\text{radius})^2 = \pi[R(x)]^2.$$

So the definition of volume gives :

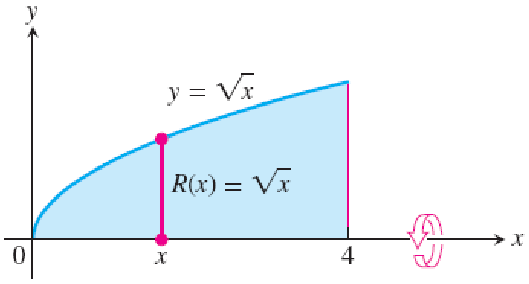
$$V = \int_a^b A(x)\,dx = \int_a^b \pi[R(x)]^2\,dx$$

A Solids Volume (Rotation About the *x*-Axis) :

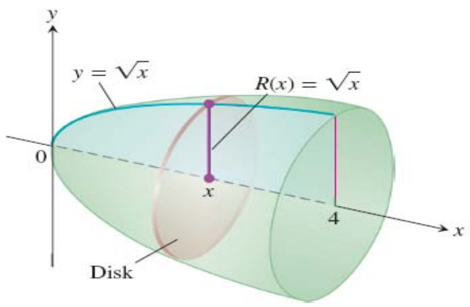
EX :

The region between the curve $y = \sqrt{x}, 0 \le x \le 4$ and the x-axis is revolved about the x-axis to generate a solid. Find its volume.

Sol: We draw figures showing the region, a typical radius, and the generated solid .



(a)



The volume is

$$V = \int_a^b \pi[R(x)]^2\,dx$$

$$= \int_0^4 \pi\left[\sqrt{x}\right]^2 dx \qquad R(x) = \sqrt{x}$$

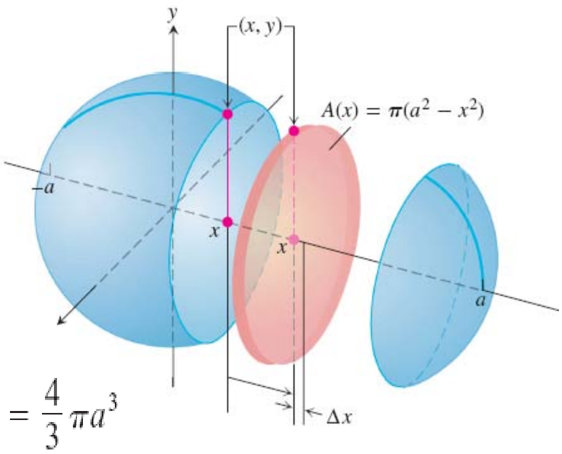
$$= \pi\int_0^4 x\,dx = \pi\frac{x^2}{2}\Big]_0^4 = \pi\frac{(4)^2}{2} = 8\pi.$$

Volume of a Sphere :

EX :

The circle $x^2 + y^2 = a$ is rotated about the *x*-axis to generate a sphere. Find its volume.

SOL :



$$A(x) = \pi y^2 = \pi(a^2 - x^2)$$

Therefore, the volume is

$$V = \int_{-a}^{a} A(x)\,dx = \int_{-a}^{a} \pi(a^2 - x^2)\,dx = \pi\left[a^2x - \frac{x^3}{3}\right]_{-a}^{a} = \frac{4}{3}\pi a^3$$

EX : Find the volumes of the solids generated by revolving the regions bounded by the lines and curves about the x-axis.

$y = x^2, \quad y = 0, \quad x = 2$

$y = \sqrt{9 - x^2}, \quad y = 0$

$y = x^3, \quad y = 0, \quad x = 2$

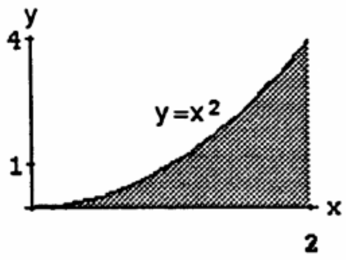
$y = x - x^2, \quad y = 0$

$y = \sqrt{\cos x}, \quad 0 \le x \le \pi/2, \quad y = 0, \quad x = 0$

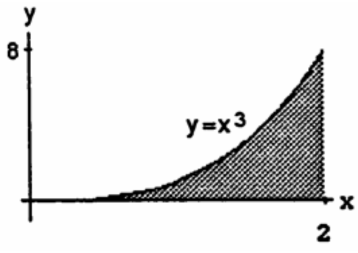
$y = \sec x, \quad y = 0, \quad x = -\pi/4, \quad x = \pi/4$

SOL :

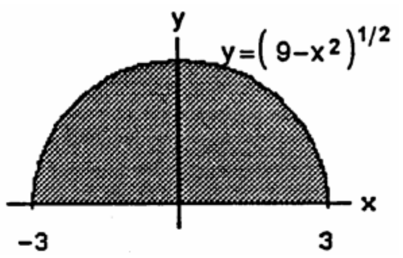
$$R(x) = x^2 \;\Rightarrow\; V = \int_0^2 \pi[R(x)]^2\,dx = \pi \int_0^2 (x^2)^2\,dx$$
$$= \pi \int_0^2 x^4\,dx = \pi\left[\frac{x^5}{5}\right]_0^2 = \frac{32\pi}{5}$$



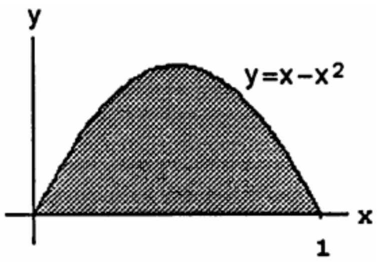
$$R(x) = x^3 \;\Rightarrow\; V = \int_0^2 \pi[R(x)]^2\,dx = \pi\int_0^2 (x^3)^2\,dx$$
$$= \pi \int_0^2 x^6\,dx = \pi\left[\frac{x^7}{7}\right]_0^2 = \frac{128\pi}{7}$$



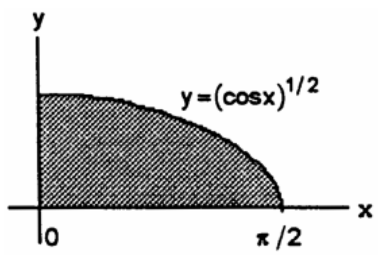
$$R(x) = \sqrt{9-x^2} \;\Rightarrow\; V = \int_{-3}^3 \pi[R(x)]^2\,dx = \pi\int_{-3}^3 (9-x^2)\,dx$$
$$= \pi\left[9x - \frac{x^3}{3}\right]_{-3}^3 = 2\pi\left[9(3) - \frac{27}{3}\right] = 2\cdot\pi\cdot 18 = 36\pi$$



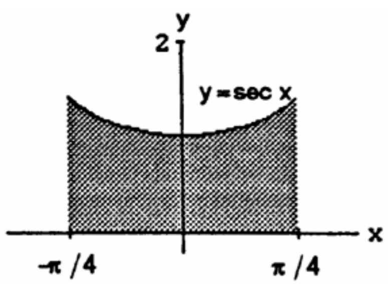
$$R(x) = x - x^2 \;\Rightarrow\; V = \int_0^1 \pi[R(x)]^2\,dx = \pi\int_0^1 (x-x^2)^2\,dx$$
$$= \pi\int_0^1 (x^2 - 2x^3 + x^4)\,dx = \pi\left[\frac{x^3}{3} - \frac{2x^4}{4} + \frac{x^5}{5}\right]_0^1$$
$$= \pi\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right) = \frac{\pi}{30}(10 - 15 + 6) = \frac{\pi}{30}$$



$$R(x) = \sqrt{\cos x} \;\Rightarrow\; V = \int_0^{\pi/2} \pi[R(x)]^2\,dx = \pi\int_0^{\pi/2} \cos x\,dx$$
$$= \pi\,[\sin x]_0^{\pi/2} = \pi(1-0) = \pi$$



$$R(x) = \sec x \;\Rightarrow\; V = \int_{-\pi/4}^{\pi/4} \pi[R(x)]^2\,dx = \pi\int_{-\pi/4}^{\pi/4} \sec^2 x\,dx$$
$$= \pi\,[\tan x]_{-\pi/4}^{\pi/4} = \pi[1-(-1)] = 2\pi$$



EX : Find the volumes of the solids generated by revolving the regions bounded by the lines and curves about the x-axis.

$y = x, \quad y = 1, \quad x = 0$

$y = x^2 + 1, \quad y = x + 3$

$y = 2\sqrt{x}, \quad y = 2, \quad x = 0$

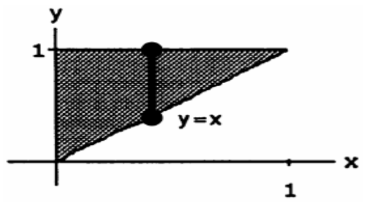
$y = 4 - x^2, \quad y = 2 - x$

$y = \sec x, \quad y = \sqrt{2}, \quad -\pi/4 \leq x \leq \pi/4$

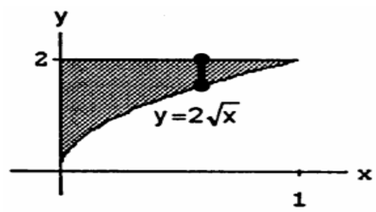
$y = \sec x, \quad y = \tan x, \quad x = 0, \quad x = 1$

SOL :

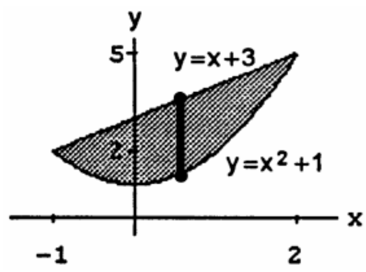
$r(x) = x$ and $R(x) = 1 \Rightarrow V = \int_0^1 \pi\left([R(x)]^2 - [r(x)]^2\right) dx$
$= \int_0^1 \pi\left(1 - x^2\right) dx = \pi\left[x - \frac{x^3}{3}\right]_0^1 = \pi\left[\left(1 - \frac{1}{3}\right) - 0\right] = \frac{2\pi}{3}$



$r(x) = 2\sqrt{x}$ and $R(x) = 2 \Rightarrow V = \int_0^1 \pi\left([R(x)]^2 - [r(x)]^2\right) dx$
$= \pi\int_0^1 (4 - 4x)\, dx = 4\pi\left[x - \frac{x^2}{2}\right]_0^1 = 4\pi\left(1 - \frac{1}{2}\right) = 2\pi$

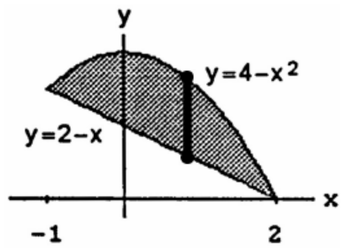


$r(x) = x^2 + 1$ and $R(x) = x + 3$



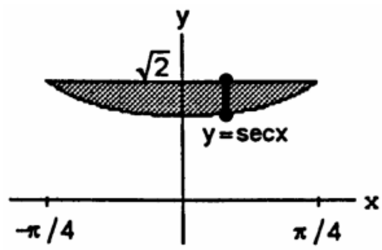
$$\Rightarrow V = \int_{-1}^{2} \pi\left([R(x)]^2 - [r(x)]^2\right) dx$$
$$= \pi \int_{-1}^{2} \left[(x+3)^2 - \left(x^2+1\right)^2\right] dx$$
$$= \pi \int_{-1}^{2} \left[(x^2+6x+9) - (x^4+2x^2+1)\right] dx$$
$$= \pi \int_{-1}^{2} \left(-x^4 - x^2 + 6x + 8\right) dx$$
$$= \pi \left[-\frac{x^5}{5} - \frac{x^3}{3} + \frac{6x^2}{2} + 8x\right]_{-1}^{2}$$
$$= \pi\left[\left(-\frac{32}{5} - \frac{8}{3} + \frac{24}{2} + 16\right) - \left(\frac{1}{5} + \frac{1}{3} + \frac{6}{2} - 8\right)\right] = \pi\left(-\frac{33}{5} - 3 + 28 - 3 + 8\right) = \pi\left(\frac{5\cdot 30 - 33}{5}\right) = \frac{117\pi}{5}$$

$r(x) = 2 - x$ and $R(x) = 4 - x^2$



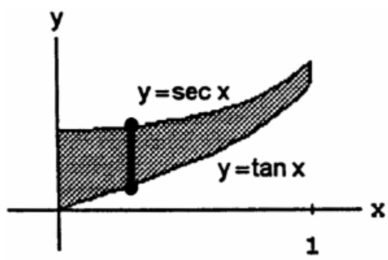
$$\Rightarrow V = \int_{-1}^{2} \pi\left([R(x)]^2 - [r(x)]^2\right) dx$$
$$= \pi \int_{-1}^{2} \left[\left(4 - x^2\right)^2 - (2-x)^2\right] dx$$
$$= \pi \int_{-1}^{2} \left[(16 - 8x^2 + x^4) - (4 - 4x + x^2)\right] dx$$
$$= \pi \int_{-1}^{2} \left(12 + 4x - 9x^2 + x^4\right) dx$$
$$= \pi \left[12x + 2x^2 - 3x^3 + \frac{x^5}{5}\right]_{-1}^{2}$$
$$= \pi\left[\left(24 + 8 - 24 + \frac{32}{5}\right) - \left(-12 + 2 + 3 - \frac{1}{5}\right)\right] = \pi\left(15 + \frac{33}{5}\right) = \frac{108\pi}{5}$$

$r(x) = \sec x$ and $R(x) = \sqrt{2}$



$$\Rightarrow V = \int_{-\pi/4}^{\pi/4} \pi\left([R(x)]^2 - [r(x)]^2\right) dx$$
$$= \pi \int_{-\pi/4}^{\pi/4} \left(2 - \sec^2 x\right) dx = \pi[2x - \tan x]_{-\pi/4}^{\pi/4}$$
$$= \pi\left[\left(\frac{\pi}{2} - 1\right) - \left(-\frac{\pi}{2} + 1\right)\right] = \pi(\pi - 2)$$

$R(x) = \sec x$ and $r(x) = \tan x$



$$\Rightarrow V = \int_{0}^{1} \pi\left([R(x)]^2 - [r(x)]^2\right) dx$$
$$= \pi \int_{0}^{1} \left(\sec^2 x - \tan^2 x\right) dx = \pi \int_{0}^{1} 1\, dx = \pi[x]_0^1 = \pi$$