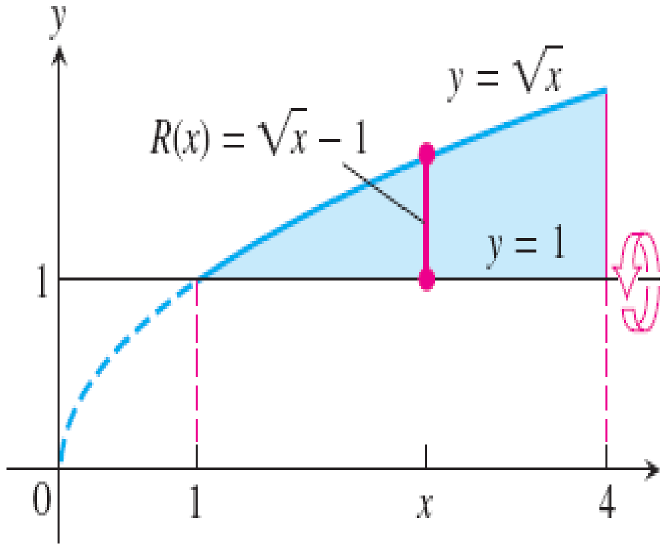
A Solids Volume (Rotation About the Line y = 1) :

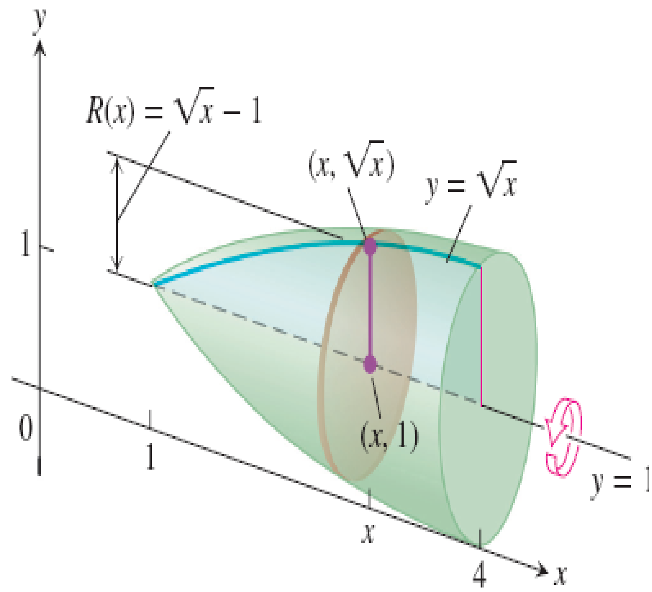
EX :

Find the volume of the solid generated by revolving the region bounded by

$y = \sqrt{x}$ **and the line y = 1 , x = 4 , about the line y = 1 .**

Sol: We draw figures showing the region, a typical radius, and the generated solid .





$$V = \int_1^4 \pi[R(x)]^2\,dx$$

$$= \int_1^4 \pi\left[\sqrt{x} - 1\right]^2 dx$$

$$= \pi \int_1^4 \left[x - 2\sqrt{x} + 1\right] dx$$

$$= \pi\left[\frac{x^2}{2} - 2 \cdot \frac{2}{3} x^{3/2} + x\right]_1^4 = \frac{7\pi}{6}.$$

EX :

Find the volume of the solid generated by revolving the triangular region bounded by the lines $y = 2x$, $y = 0$, and $x = 1$ about

a. the line $x = 1$. **b.** the line $x = 2$.

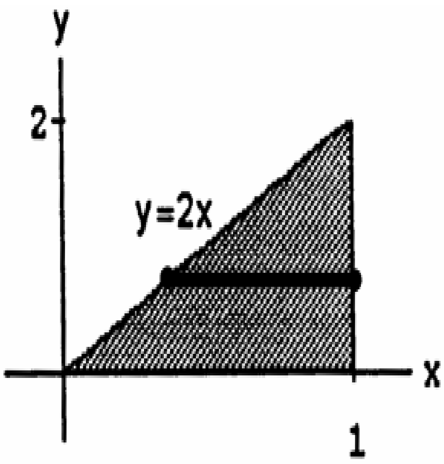
SOL :

(a) $r(y) = 0$ and $R(y) = 1 - \frac{y}{2}$

$$\Rightarrow V = \int_0^2 \pi\left([R(y)]^2 - [r(y)]^2\right) dy$$

$$= \pi \int_0^2 \left(1 - \frac{y}{2}\right)^2 dy = \pi \int_0^2 \left(1 - y + \frac{y^2}{4}\right) dy$$

$$= \pi \left[y - \frac{y^2}{2} + \frac{y^3}{12}\right]_0^2 = \pi\left(2 - \frac{4}{2} + \frac{8}{12}\right) = \frac{2\pi}{3}$$



(b) $r(y) = 1$ and $R(y) = 2 - \frac{y}{2}$

$$\Rightarrow V = \int_0^2 \pi\left([R(y)]^2 - [r(y)]^2\right) dy = \pi \int_0^2 \left[\left(2 - \frac{y}{2}\right)^2 - 1\right] dy = \pi \int_0^2 \left(4 - 2y + \frac{y^2}{4} - 1\right) dy$$

$$= \pi \int_0^2 \left(3 - 2y + \frac{y^2}{4}\right) dy = \pi\left[3y - y^2 + \frac{y^3}{12}\right]_0^2 = \pi\left(6 - 4 + \frac{8}{12}\right) = \pi\left(2 + \frac{2}{3}\right) = \frac{8\pi}{3}$$

EX :

Find the volume of the solid generated by revolving the region bounded by the parabola $y = x^2$ and the line $y = 1$ about

a. the line $y = 1$. **b.** the line $y = 2$.

c. the line $y = -1$.

SOL :

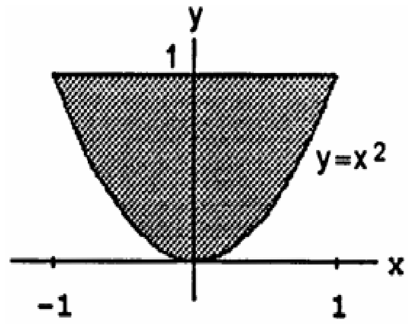
(a) $r(x) = 0$ and $R(x) = 1 - x^2$

$$\Rightarrow V = \int_{-1}^{1} \pi\left([R(x)]^2 - [r(x)]^2\right) dx$$

$$= \pi \int_{-1}^{1} (1 - x^2)^2\, dx = \pi \int_{-1}^{1} (1 - 2x^2 + x^4)\, dx$$

$$= \pi \left[x - \frac{2x^3}{3} + \frac{x^5}{5}\right]_{-1}^{1} = 2\pi\left(1 - \frac{2}{3} + \frac{1}{5}\right)$$

$$= 2\pi\left(\frac{15-10+3}{15}\right) = \frac{16\pi}{15}$$



(b) $r(x) = 1$ and $R(x) = 2 - x^2 \Rightarrow V = \int_{-1}^{1} \pi\left([R(x)]^2 - [r(x)]^2\right) dx = \pi \int_{-1}^{1} \left[(2 - x^2)^2 - 1\right] dx$

$$= \pi \int_{-1}^{1} (4 - 4x^2 + x^4 - 1)\, dx = \pi \int_{-1}^{1} (3 - 4x^2 + x^4)\, dx = \pi \left[3x - \frac{4}{3}x^3 + \frac{x^5}{5}\right]_{-1}^{1} = 2\pi\left(3 - \frac{4}{3} + \frac{1}{5}\right)$$

$$= \frac{2\pi}{15}(45 - 20 + 3) = \frac{56\pi}{15}$$

(c) $r(x) = 1 + x^2$ and $R(x) = 2 \Rightarrow V = \int_{-1}^{1} \pi\left([R(x)]^2 - [r(x)]^2\right) dx = \pi \int_{-1}^{1} \left[4 - (1 + x^2)^2\right] dx$

$$= \pi \int_{-1}^{1} (4 - 1 - 2x^2 - x^4)\, dx = \pi \int_{-1}^{1} (3 - 2x^2 - x^4)\, dx = \pi \left[3x - \frac{2}{3}x^3 - \frac{x^5}{5}\right]_{-1}^{1} = 2\pi\left(3 - \frac{2}{3} - \frac{1}{5}\right)$$

$$= \frac{2\pi}{15}(45 - 10 - 3) = \frac{64\pi}{15}$$

EX :

By integration, find the volume of the solid generated by revolving the triangular region with vertices (0, 0), (b, 0), (0, h) about

a. the x-axis. **b.** the y-axis.

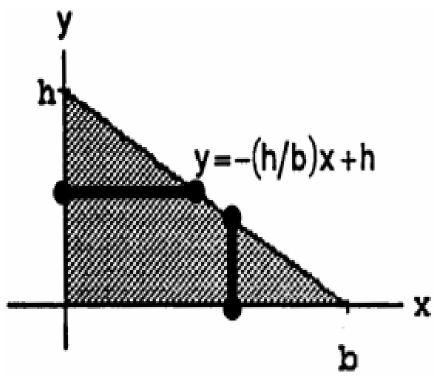
(a) $r(x) = 0$ and $R(x) = -\frac{h}{b}x + h$

$$\Rightarrow V = \int_{0}^{b} \pi\left([R(x)]^2 - [r(x)]^2\right) dx$$

$$= \pi \int_{0}^{b} \left(-\frac{h}{b}x + h\right)^2 dx$$

$$= \pi \int_{0}^{b} \left(\frac{h^2}{b^2}x^2 - \frac{2h^2}{b}x + h^2\right) dx$$

$$= \pi h^2 \left[\frac{x^3}{3b^2} - \frac{x^2}{b} + x\right]_{0}^{b} = \pi h^2\left(\frac{b}{3} - b + b\right) = \frac{\pi h^2 b}{3}$$



(b) $r(y) = 0$ and $R(y) = b\left(1 - \frac{y}{h}\right) \Rightarrow V = \int_{0}^{h} \pi\left([R(y)]^2 - [r(y)]^2\right) dy = \pi b^2 \int_{0}^{h} \left(1 - \frac{y}{h}\right)^2 dy$

$$= \pi b^2 \int_{0}^{h} \left(1 - \frac{2y}{h} + \frac{y^2}{h^2}\right) dy = \pi b^2 \left[y - \frac{y^2}{h} + \frac{y^3}{3h^2}\right]_{0}^{h} = \pi b^2\left(h - h + \frac{h}{3}\right) = \frac{\pi b^2 h}{3}$$