

EXAMPLES : Evaluate the following integrals by using trigonometric substitutions :

$$17. \int \frac{8 dw}{w^2 \sqrt{4 - w^2}}$$

$$18. \int \frac{\sqrt{9 - w^2}}{w^2} dw$$

$$19. \int_0^{\sqrt{3}/2} \frac{4x^2 dx}{(1 - x^2)^{3/2}}$$

$$20. \int_0^1 \frac{dx}{(4 - x^2)^{3/2}}$$

$$21. \int \frac{dx}{(x^2 - 1)^{3/2}}, \quad x > 1$$

$$22. \int \frac{x^2 dx}{(x^2 - 1)^{5/2}}, \quad x > 1$$

$$23. \int \frac{(1 - x^2)^{3/2}}{x^6} dx$$

$$24. \int \frac{(1 - x^2)^{1/2}}{x^4} dx$$

$$25. \int \frac{8 dx}{(4x^2 + 1)^2}$$

$$26. \int \frac{6 dt}{(9t^2 + 1)^2}$$

$$27. \int \frac{v^2 dv}{(1 - v^2)^{5/2}}$$

$$28. \int \frac{(1 - r^2)^{5/2}}{r^8} dr$$

$$29. \int_0^{\ln 4} \frac{e^t dt}{\sqrt{e^{2t} + 9}}$$

$$30. \int_{\ln(3/4)}^{\ln(4/3)} \frac{e^t dt}{(1 + e^{2t})^{3/2}}$$

$$31. \int_{1/12}^{1/4} \frac{2 dt}{\sqrt{t} + 4t\sqrt{t}}$$

$$32. \int_1^e \frac{dy}{y\sqrt{1 + (\ln y)^2}}$$

$$33. \int \frac{dx}{x\sqrt{x^2 - 1}}$$

$$34. \int \frac{dx}{1 + x^2}$$

$$35. \int \frac{x dx}{\sqrt{x^2 - 1}}$$

$$36. \int \frac{dx}{\sqrt{1 - x^2}}$$

SOLUTIONS :

$$17. w = 2 \sin \theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}, \quad dw = 2 \cos \theta d\theta, \quad \sqrt{4 - w^2} = 2 \cos \theta;$$

$$\int \frac{8 dw}{w^2 \sqrt{4 - w^2}} = \int \frac{8 \cdot 2 \cos \theta d\theta}{4 \sin^2 \theta \cdot 2 \cos \theta} = 2 \int \frac{d\theta}{\sin^2 \theta} = -2 \cot \theta + C = \frac{-2\sqrt{4 - w^2}}{w} + C$$

$$18. w = 3 \sin \theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}, \quad dw = 3 \cos \theta d\theta, \quad \sqrt{9 - w^2} = 3 \cos \theta;$$

$$\int \frac{\sqrt{9 - w^2}}{w^2} dw = \int \frac{3 \cos \theta \cdot 3 \cos \theta d\theta}{9 \sin^2 \theta} = \int \cot^2 \theta d\theta = \int \left(\frac{1 - \sin^2 \theta}{\sin^2 \theta} \right) d\theta = \int (\csc^2 \theta - 1) d\theta$$

$$= -\cot \theta - \theta + C = -\frac{\sqrt{9 - w^2}}{w} - \sin^{-1} \left(\frac{w}{3} \right) + C$$

$$19. x = \sin \theta, \quad 0 \leq \theta \leq \frac{\pi}{3}, \quad dx = \cos \theta d\theta, \quad (1 - x^2)^{3/2} = \cos^3 \theta;$$

$$\int_0^{\sqrt{3}/2} \frac{4x^2 dx}{(1 - x^2)^{3/2}} = \int_0^{\pi/3} \frac{4 \sin^2 \theta \cos \theta d\theta}{\cos^3 \theta} = 4 \int_0^{\pi/3} \left(\frac{1 - \cos^2 \theta}{\cos^2 \theta} \right) d\theta = 4 \int_0^{\pi/3} (\sec^2 \theta - 1) d\theta$$

$$= 4 [\tan \theta - \theta]_0^{\pi/3} = 4\sqrt{3} - \frac{4\pi}{3}$$

20. $x = 2 \sin \theta, 0 \leq \theta \leq \frac{\pi}{6}, dx = 2 \cos \theta d\theta, (4 - x^2)^{3/2} = 8 \cos^3 \theta;$

$$\int_0^1 \frac{dx}{(4 - x^2)^{3/2}} = \int_0^{\pi/6} \frac{2 \cos \theta d\theta}{8 \cos^3 \theta} = \frac{1}{4} \int_0^{\pi/6} \frac{d\theta}{\cos^2 \theta} = \frac{1}{4} [\tan \theta]_0^{\pi/6} = \frac{\sqrt{3}}{12} = \frac{1}{4\sqrt{3}}$$
21. $x = \sec \theta, 0 < \theta < \frac{\pi}{2}, dx = \sec \theta \tan \theta d\theta, (x^2 - 1)^{3/2} = \tan^3 \theta;$

$$\int \frac{dx}{(x^2 - 1)^{3/2}} = \int \frac{\sec \theta \tan \theta d\theta}{\tan^3 \theta} = \int \frac{\cos \theta d\theta}{\sin^2 \theta} = -\frac{1}{\sin \theta} + C = -\frac{x}{\sqrt{x^2 - 1}} + C$$
22. $x = \sec \theta, 0 < \theta < \frac{\pi}{2}, dx = \sec \theta \tan \theta d\theta, (x^2 - 1)^{5/2} = \tan^5 \theta;$

$$\int \frac{x^2 dx}{(x^2 - 1)^{5/2}} = \int \frac{\sec^2 \theta \cdot \sec \theta \tan \theta d\theta}{\tan^5 \theta} = \int \frac{\cos \theta}{\sin^4 \theta} d\theta = -\frac{1}{3 \sin^3 \theta} + C = -\frac{x^3}{3(x^2 - 1)^{3/2}} + C$$
23. $x = \sin \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}, dx = \cos \theta d\theta, (1 - x^2)^{3/2} = \cos^3 \theta;$

$$\int \frac{(1 - x^2)^{3/2} dx}{x^6} = \int \frac{\cos^3 \theta \cdot \cos \theta d\theta}{\sin^6 \theta} = \int \cot^4 \theta \csc^2 \theta d\theta = -\frac{\cot^5 \theta}{5} + C = -\frac{1}{5} \left(\frac{\sqrt{1 - x^2}}{x} \right)^5 + C$$
24. $x = \sin \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}, dx = \cos \theta d\theta, (1 - x^2)^{1/2} = \cos \theta;$

$$\int \frac{(1 - x^2)^{1/2} dx}{x^4} = \int \frac{\cos \theta \cdot \cos \theta d\theta}{\sin^4 \theta} = \int \cot^2 \theta \csc^2 \theta d\theta = -\frac{\cot^3 \theta}{3} + C = -\frac{1}{3} \left(\frac{\sqrt{1 - x^2}}{x} \right)^3 + C$$
25. $x = \frac{1}{2} \tan \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}, dx = \frac{1}{2} \sec^2 \theta d\theta, (4x^2 + 1)^2 = \sec^4 \theta;$

$$\int \frac{8 dx}{(4x^2 + 1)^2} = \int \frac{8 \left(\frac{1}{2} \sec^2 \theta \right) d\theta}{\sec^4 \theta} = 4 \int \cos^2 \theta d\theta = 2(\theta + \sin \theta \cos \theta) + C = 2 \tan^{-1} 2x + \frac{4x}{(4x^2 + 1)} + C$$
26. $t = \frac{1}{3} \tan \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}, dt = \frac{1}{3} \sec^2 \theta d\theta, 9t^2 + 1 = \sec^2 \theta;$

$$\int \frac{6 dt}{(9t^2 + 1)^2} = \int \frac{6 \left(\frac{1}{3} \sec^2 \theta \right) d\theta}{\sec^4 \theta} = 2 \int \cos^2 \theta d\theta = \theta + \sin \theta \cos \theta + C = \tan^{-1} 3t + \frac{3t}{(9t^2 + 1)} + C$$
27. $v = \sin \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}, dv = \cos \theta d\theta, (1 - v^2)^{5/2} = \cos^5 \theta;$

$$\int \frac{v^2 dv}{(1 - v^2)^{5/2}} = \int \frac{\sin^2 \theta \cos \theta d\theta}{\cos^5 \theta} = \int \tan^2 \theta \sec^2 \theta d\theta = \frac{\tan^3 \theta}{3} + C = \frac{1}{3} \left(\frac{v}{\sqrt{1 - v^2}} \right)^3 + C$$
28. $r = \sin \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2};$

$$\int \frac{(1 - r^2)^{5/2} dr}{r^8} = \int \frac{\cos^5 \theta \cdot \cos \theta d\theta}{\sin^8 \theta} = \int \cot^6 \theta \csc^2 \theta d\theta = -\frac{\cot^7 \theta}{7} + C = -\frac{1}{7} \left[\frac{\sqrt{1 - r^2}}{r} \right]^7 + C$$
29. Let $e^t = 3 \tan \theta, t = \ln(3 \tan \theta), \tan^{-1} \left(\frac{1}{3} \right) \leq \theta \leq \tan^{-1} \left(\frac{4}{3} \right), dt = \frac{\sec^2 \theta}{\tan \theta} d\theta, \sqrt{e^{2t} + 9} = \sqrt{9 \tan^2 \theta + 9} = 3 \sec \theta;$

$$\int_0^{\ln 4} \frac{e^t dt}{\sqrt{e^{2t} + 9}} = \int_{\tan^{-1}(1/3)}^{\tan^{-1}(4/3)} \frac{3 \tan \theta \cdot \sec^2 \theta d\theta}{\tan \theta \cdot 3 \sec \theta} = \int_{\tan^{-1}(1/3)}^{\tan^{-1}(4/3)} \sec \theta d\theta = [\ln |\sec \theta + \tan \theta|]_{\tan^{-1}(1/3)}^{\tan^{-1}(4/3)}$$

$$= \ln \left(\frac{5}{3} + \frac{4}{3} \right) - \ln \left(\frac{\sqrt{10}}{3} + \frac{1}{3} \right) = \ln 9 - \ln \left(1 + \sqrt{10} \right)$$
30. Let $e^t = \tan \theta, t = \ln(\tan \theta), \tan^{-1} \left(\frac{3}{4} \right) \leq \theta \leq \tan^{-1} \left(\frac{4}{3} \right), dt = \frac{\sec^2 \theta}{\tan \theta} d\theta, 1 + e^{2t} = 1 + \tan^2 \theta = \sec^2 \theta;$

$$\int_{\ln(3/4)}^{\ln(4/3)} \frac{e^t dt}{(1 + e^{2t})^{3/2}} = \int_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)} \frac{(\tan \theta) \left(\frac{\sec^2 \theta}{\tan \theta} \right) d\theta}{\sec^3 \theta} = \int_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)} \cos \theta d\theta = [\sin \theta]_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)} = \frac{4}{5} - \frac{3}{5} = \frac{1}{5}$$
31. $\int_{1/\sqrt{3}}^{1/4} \frac{2 dt}{\sqrt{t+4t\sqrt{t}}}; [u = 2\sqrt{t}, du = \frac{1}{\sqrt{t}} dt] \rightarrow \int_{1/\sqrt{3}}^{1/4} \frac{2 du}{1 - u^2}; u = \tan \theta, \frac{\pi}{6} \leq \theta \leq \frac{\pi}{4}, du = \sec^2 \theta d\theta, 1 + u^2 = \sec^2 \theta;$

$$\int_{1/\sqrt{3}}^{1/4} \frac{2 du}{1 - u^2} = \int_{\pi/6}^{\pi/4} \frac{2 \sec^2 \theta d\theta}{\sec^2 \theta} = [2\theta]_{\pi/6}^{\pi/4} = 2 \left(\frac{\pi}{4} - \frac{\pi}{6} \right) = \frac{\pi}{6}$$

32. $y = e^{\tan \theta}$, $0 \leq \theta \leq \frac{\pi}{4}$, $dy = e^{\tan \theta} \sec^2 \theta d\theta$, $\sqrt{1 + (\ln y)^2} = \sqrt{1 + \tan^2 \theta} = \sec \theta$;
$$\int_1^e \frac{dy}{y\sqrt{1 + (\ln y)^2}} = \int_0^{\pi/4} \frac{e^{\tan \theta} \sec^2 \theta}{e^{\tan \theta} \sec \theta} d\theta = \int_0^{\pi/4} \sec \theta d\theta = [\ln |\sec \theta + \tan \theta|]_0^{\pi/4} = \ln(1 + \sqrt{2})$$

33. $x = \sec \theta$, $0 < \theta < \frac{\pi}{2}$, $dx = \sec \theta \tan \theta d\theta$, $\sqrt{x^2 - 1} = \sqrt{\sec^2 \theta - 1} = \tan \theta$;
$$\int \frac{dx}{x\sqrt{x^2 - 1}} = \int \frac{\sec \theta \tan \theta d\theta}{\sec \theta \tan \theta} = \theta + C = \sec^{-1} x + C$$

34. $x = \tan \theta$, $dx = \sec^2 \theta d\theta$, $1 + x^2 = \sec^2 \theta$;
$$\int \frac{dx}{x^2 + 1} = \int \frac{\sec^2 \theta d\theta}{\sec^2 \theta} = \theta + C = \tan^{-1} x + C$$

35. $x = \sec \theta$, $dx = \sec \theta \tan \theta d\theta$, $\sqrt{x^2 - 1} = \sqrt{\sec^2 \theta - 1} = \tan \theta$;
$$\int \frac{x dx}{\sqrt{x^2 - 1}} = \int \frac{\sec \theta \cdot \sec \theta \tan \theta d\theta}{\tan \theta} = \int \sec^2 \theta d\theta = \tan \theta + C = \sqrt{x^2 - 1} + C$$

36. $x = \sin \theta$, $dx = \cos \theta d\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$;
$$\int \frac{dx}{\sqrt{1 - x^2}} = \int \frac{\cos \theta d\theta}{\cos \theta} = \theta + C = \sin^{-1} x + C$$