***Example 1***

The derivative of a function $f(x)$ at a particular value of $x$ can be approximately calculated by

$$f'(x) \approx \frac{f(x+h)-f(x)}{h}$$

of $f'(2)$ For $f(x)=7e^{0.5x}$ and $h=0.3$, find

a) the approximate value of $f'(2)$

b) the true value of $f'(2)$

c) the true error for part (a)

d) the relative true error at $x=2$.

**Solution:**

a) $$f'(x) \approx \frac{f(x+h)-f(x)}{h}$$

For $x=2$ and $h=0.3$,

$$f'(2) \approx \frac{f(2+0.3)-f(2)}{0.3}$$

$$= \frac{f(2.3)-f(2)}{0.3}$$

$$= \frac{7e^{0.5(2.3)}-7e^{0.5(2)}}{0.3}$$

$$= \frac{22.107-19.028}{0.3}$$

$$= 10.265$$

b) The exact value of $f'(2)$ can be calculated by using our knowledge of differential calculus.

$$f(x)=7e^{0.5x}$$

$$f'(x)=7\times 0.5\times e^{0.5x}$$

$$=3.5e^{0.5x}$$

So the true value of $f'(2)$ is

$$f'(2)=3.5e^{0.5(2)}$$

$$=9.5140$$

c) True error is calculated as

$$e_x = |\text{True value} - \text{Approximate value}|$$

$$=|9.5140-10.265|$$

$$=|-0.7561| = 0.7561$$

d) $$\delta_x = \frac{Absoluete\ Value}{|true\ Value\,|}$$

$$= \frac{0.7561}{|9.5140|}$$

$$= 0.0758895$$

$= 7.58895\%$

## 8 – **Approximate Error:-**

is denoted by $E_a$ and is defined as the difference between the present approximation and previous approximation.

Approximate Error= $|\text{Present Approximation} - \text{Previous Approximation}|$

**Relative approximate error**:-

is denoted by $\boldsymbol{\delta_a}$ and is defined as the ratio between the approximate error and the present approximation.

$$\text{Relative approximate error} = \frac{\text{approximate error}}{|\text{Present Approximation}|}$$

***Example 3***

The derivative of a function $f(x)$ at a particular value of $x$ can be approximately calculated by

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

For $f(x) = 7e^{0.5x}$ and at $x = 2$, find the following

a) $f'(2)$ using $h = 0.3$

b) $f'(2)$ using $h = 0.15$

c) approximate error for the value of $f'(2)$ for part (b)

d) the relative approximate error

**Solution:**

a) The approximate expression for the derivative of a function is

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}.$$

For $x = 2$ and $h = 0.3$,

$$f'(2) \approx \frac{f(2+0.3) - f(2)}{0.3}$$

$$= \frac{f(2.3) - f(2)}{0.3}$$

$$= \frac{7e^{0.5(2.3)} - 7e^{0.5(2)}}{0.3}$$

$$= \frac{22.107 - 19.028}{0.3}$$

$$= 10.265$$

b) Repeat the procedure of part (a) with $h = 0.15$,

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

For $x = 2$ and $h = 0.15$,

$$f'(2) \approx \frac{f(2+0.15)-f(2)}{0.15}$$
$$= \frac{f(2.15)-f(2)}{0.15}$$
$$= \frac{7e^{0.5(2.15)}-7e^{0.5(2)}}{0.15}$$
$$= \frac{20.50-19.028}{0.15}$$
$$= 9.8799$$

c) So the approximate error , $E_a$ is

$$E_a = |\text{True value} - \text{Approximate value}|$$
$$= |9.8799 - 10.265|$$
$$= |-0.38474| = 0.38474$$

d) Relative approximate error $= \frac{\text{approximate error}}{|\text{Present Approximation}|}$

$$= \frac{0.38474}{|9.8799|}$$

= 0.038942*100%

= 3.8942%

**Q/** While solving a mathematical model using numerical methods, how can we use relative approximate errors to minimize the error?
*A:* In a numerical method that uses iterative methods, a user can calculate relative approximate error $\delta_a$ at the end of each iteration. The user may pre-specify a minimum acceptable tolerance called the **pre-specified tolerance** $\boldsymbol{\delta_s}$ . If the absolute relative approximate error $\delta_a$ is less than or equal to the pre-specified tolerance $\delta_s$ , that is, $\delta_a \le \delta_s$ , then the acceptable error has been reached and no more iterations would be required . Alternatively, one may pre-specify how many significant digits they would like to be correct in their answer. In that case, if one wants at least $m$ significant digits to be correct in the answer, then you would need to have the absolute relative approximate error.
$\delta_a \le 0.5 * 10^{2-m}$ %

**Example 5**

If one chooses 6 terms of the Maclourin series for $e^x$ to calculate $e^{0.7}$ , how many significant digits can you trust in the solution? Find your answer without knowing or using the exact answer.

**Solution**

$$e^x = 1 + x + \frac{x^2}{2!} + \dots\dots\dots\dots\dots$$

Using 6 terms, we get the current approximation as

$$e^{0.7} \cong 1 + 0.7 + \frac{0.7^2}{2!} + \frac{0.7^3}{3!} + \frac{0.7^4}{4!} + \frac{0.7^5}{5!}$$

$$= 2.0136$$

Using 5 terms, we get the previous approximation as

$$e^{0.7} \cong 1 + 0.7 + \frac{0.7^2}{2!} + \frac{0.7^3}{3!} + \frac{0.7^4}{4!}$$

$$= 2.0122$$

The percentage absolute relative approximate error is

$$\delta_a = \frac{|2.0136 - 2.0122|}{|2.0136|} * 100$$

$$= 0.069527\ \%$$

Since $\delta_a \le 0.5 * 10^{2-2}\%$ , at least 2 significant digits are correct in the answer of

$$e^{0.7} \cong 2.0136$$

**Example 6:-**

Find the Taylor Series expansion for $\sin(2)$ at $x_0 = \frac{\pi}{2}$

**Sol:**

$f(x) = sinx$ $\qquad f\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1$

$f'(x) = \cos(x) = \sin(x + \frac{\pi}{2})$ $\qquad f'\left(\frac{\pi}{2}\right) = 0$

$f''(x) = -\sin(x) = \sin(x + \pi)$ $\qquad f''\left(\frac{\pi}{2}\right) = -1$

$f'''(\mathrm{x}) = -\cos(\mathrm{x}) = \sin\left(\mathrm{x} + \frac{3\pi}{2}\right)$ $\qquad f'''\left(\frac{\pi}{2}\right) = 0$

$$\sin(x) = 1 + 0\left(x - \frac{\pi}{2}\right) - \frac{1\left(x - \frac{\pi}{2}\right)^2}{2!} + \frac{0\left(x - \frac{\pi}{2}\right)^3}{3!} + \frac{1\left(x - \frac{\pi}{2}\right)^4}{3!} + \cdots$$

$$\therefore \sin(2) = 1 + 0(0.42920) + \frac{1(0.42920)^2}{2!} + \frac{0(0.42920)^3}{3!} + \frac{1(0.42920)^4}{4!} + \cdots$$

$$\cong\ 0.90931$$

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Can using form alternative for Taylor series now

$$f(x_0 + h) = f(x_0) + f'(x_0)h + f''(x_0)\frac{h^2}{2!} + f'''(x_0)\frac{h^3}{3!} + f''''(x_0)\frac{h^4}{4!} + \cdots$$

**Example7**

Find the value of $f(6)$ given that $f(4)=125$, $f'(4)=74$, $f''(4)=30$, $f'''(4)=6$ and all other higher derivatives of $f(x)$ at $x=4$ are zero.

**Solution**

$$f(x_0+h)=f(x_0)+f'(x_0)h+f''(x_0)\frac{h^2}{2!}+f'''(x_0)\frac{h^3}{3!}+f''''(x_0)\frac{h^4}{4!}+\cdots$$

$\because x_0=4$

$h=6-4=2$

since fourth and higher derivative of $f(x)$ are zero at $x=4$

$$f(4+2)=f(4)+f'(4)2+f''(4)\frac{2^2}{2!}+f'''(4)\frac{2^3}{3!}$$

$$f(6)=125+74(2)+30\left(\frac{2^2}{2!}\right)+6\frac{2^3}{3!}$$

$$=125+148+60+8$$

$$=341$$