Finite Sample Space with Equally Points

It is a sample space with finite number of outcomes as N, then it is called finite sample space with equally points if the probability of each its outcome is $\frac{1}{N}$. Let $\mathbf{P}[.]$ be a probability function, where P: $\boldsymbol{\mathcal{A}} \rightarrow [0,1]$ and $\boldsymbol{\mathcal{A}}$ is event space satisfies the following conditions:

1) $\mathbf{P}[\{w_1\}] = \mathbf{P}[\{w_2\}] = \cdots = \mathbf{P}[\{w_N\}]$

2) $A \in \boldsymbol{\mathcal{A}} \rightarrow \boldsymbol{P}[A] = \frac{N(A)}{N}$, where $N(A) =$ number of elements that A contains, and $\Omega = \{w_1, w_2, \dots, w_N\}$. Then it is readily checked that the set function P[·] satisfies the three axioms and hence is a probability function.

Definition 17 Equally likely probability function The probability function $P[\cdot]$ satisfying conditions (i) and (ii) above is defined to be an *equally likely probability function.* ////

EXAMPLE 14 Let Ω be the sample space corresponding to the experiment of tossing two dice, and let $\mathscr{A}$ be the collection of all subsets of Ω. For any $A \in \mathscr{A}$ define $N(A) =$ number of outcomes, or points in Ω, that are in A. Then $N(\phi) = 0$, $N(\Omega) = 36$, and $N(A) = 6$ if A is the event containing those outcomes having a total of seven spots up. ////

Finite sample space without equally likely points We saw for finite sample spaces with equally likely sample points that $P[A] = N(A)/N(\Omega)$ for any event A. For finite sample spaces without equally likely sample points, things are not quite as simple, but we can completely define the values of $P[A]$ for each of the $2^{N(\Omega)}$ events A by specifying the value of $P[\cdot]$ for each of the $N = N(\Omega)$ elementary events. Let $\Omega = \{\omega_1, \ldots, \omega_N\}$, and assume $p_j = P[\{\omega_j\}]$ for $j = 1, \ldots, N$. Since

$$1 = P[\Omega] = P\left[\bigcup_{j=1}^{N} \{\omega_j\}\right] = \sum_{j=1}^{N} P[\{\omega_j\}],$$

$$\sum_{j=1}^{N} p_j = 1.$$

For any event A, define $P[A] = \Sigma p_j$, where the summation is over those ω_j belonging to A. It can be shown that $P[\cdot]$ so defined satisfies the three axioms and hence is a probability function.

EXAMPLE 22 Consider an experiment that has N outcomes, say $\omega_1; \omega_2, \ldots, \omega_N$, where it is known that outcome ω_{j+1} is twice as likely as outcome ω_j, where $j = 1, \ldots, N-1$; that is, $p_{j+1} = 2p_j$, where $p_i = P[\{\omega_i\}]$. Find $P[A_k]$, where $A_k = \{\omega_1, \omega_2, \ldots, \omega_k\}$. Since

$$\sum_{j=1}^{N} p_j = \sum_{j=1}^{N} 2^{j-1} p_1 = p_1(1 + 2 + 2^2 + \cdots + 2^{N-1}) = p_1(2^N - 1) = 1,$$

$$p_1 = \frac{1}{2^N - 1}$$

and

$$p_j = 2^{j-1}/(2^N - 1);$$

hence

$$P[A_k] = \sum_{j=1}^{k} p_j = \sum_{j=1}^{k} 2^{j-1}/(2^N - 1) = \frac{2^k - 1}{2^N - 1}. \qquad ////$$

Conditional Probability and Independence

Definition 18 Conditional probability Let A and B be two events in $\mathscr{A}$ of the given probability space $(\Omega, \mathscr{A}, P[\cdot])$. The *conditional probability* of event A given event B, denoted by $P[A \mid B]$, is defined by

$$P[A \mid B] = \frac{P[AB]}{P[B]} \qquad \text{if} \qquad P[B] > 0, \tag{6}$$

and is left undefined if $P[B] = 0$. ////

Remark A formula that is evident from the definition is $P[AB] = P[A \mid B]P[B] = P[B \mid A]P[A]$ if both $P[A]$ and $P[B]$ are nonzero. This formula relates $P[A \mid B]$ to $P[B \mid A]$ in terms of the unconditional probabilities $P[A]$ and $P[B]$. ////

EXAMPLE 24 Consider the experiment of tossing two coins. Let $\Omega = \{(H, H), (H; T), (T, H), (T, T)\}$, and assume that each point is equally likely. Find (i) the probability of two heads given a head on the first coin and (ii) the probability of two heads given at least one head. Let $A_1 = \{\text{head on first coin}\}$ and $A_2 = \{\text{head on second coin}\}$; then the probability of two heads given a head on the first coin is

$$P[A_1 A_2 \mid A_1] = \frac{P[A_1 A_2 A_1]}{P[A_1]} = \frac{P[A_1 A_2]}{P[A_1]} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}.$$

***Howm work* 2** Does the conditional probability $\boldsymbol{P}[./B]$ satisfy the axioms of the probability f unction?

"Introduction to the Theory of Statistics"

By

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