6.2 Transforms of Derivatives and Integrals. ODEs

The Laplace transform is a method of solving ODEs and initial value problems. The crucial idea is that ***operations of calculus on functions are replaced by operations of algebra on transforms.*** Roughly, *differentiation* of $f(t)$ will correspond to *multiplication* of $\mathcal{L}(f)$ by s (see Theorems 1 and 2) and *integration* of $f(t)$ to *division* of $\mathcal{L}(f)$ by s. To solve ODEs, we must first consider the Laplace transform of derivatives.

THEOREM 1

Laplace Transform of Derivatives

The transforms of the first and second derivatives of $f(t)$ satisfy

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0) \tag{1}$$

$$\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0). \tag{2}$$

Formula (1) *holds if $f(t)$ is continuous for all $t \geqq 0$ and satisfies the growth restriction* (2) *in Sec.* 6.1 *and $f'(t)$ is piecewise continuous on every finite interval on the semi-axis $t \geqq 0$. Similarly,* (2) *holds if f and f' are continuous for all $t \geqq 0$ and satisfy the growth restriction and f'' is piecewise continuous on every finite interval on the semi-axis $t \geqq 0$.*

PROOF We prove (1) first under the *additional assumption* that f' is continuous. Then by the definition and integration by parts,

$$\mathcal{L}(f') = \int_0^\infty e^{-st} f'(t)\, dt = [e^{-st}f(t)] \Big|_0^\infty + s\int_0^\infty e^{-st} f(t)\, dt.$$

Since f satisfies (2) in Sec. 6.1, the integrated part on the right is zero at the upper limit when $s > k$, and at the lower limit it contributes $-f(0)$. The last integral is $\mathcal{L}(f)$. It exists for $s > k$ because of Theorem 3 in Sec. 6.1. Hence $\mathcal{L}(f')$ exists when $s > k$ and (1) holds.

If f' is merely piecewise continuous, the proof is similar. In this case the interval of integration of f' must be broken up into parts such that f' is continuous in each such part.

The proof of (2) now follows by applying (1) to f'' and then substituting (1), that is

$$\mathcal{L}(f'') = s\mathcal{L}(f') - f'(0) = s[s\mathcal{L}(f) - f(0)] = s^2\mathcal{L}(f) - sf(0) - f'(0). \qquad \blacksquare$$

Continuing by substitution as in the proof of (2) and using induction, we obtain the following extension of Theorem 1.

THEOREM 2

Laplace Transform of the Derivative $f^{(n)}$ of Any Order

Let $f, f', \cdots, f^{(n-1)}$ be continuous for all $t \geqq 0$ and satisfy the growth restriction (2) *in Sec.* 6.1. *Furthermore, let $f^{(n)}$ be piecewise continuous on every finite interval on the semi-axis $t \geqq 0$. Then the transform of $f^{(n)}$ satisfies*

$$\mathcal{L}(f^{(n)}) = s^n\mathcal{L}(f) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0). \tag{3}$$

EXAMPLE 1 **Transform of a Resonance Term (Sec. 2.8)**

Let $f(t) = t \sin \omega t$. Then $f(0) = 0$. $f'(t) = \sin \omega t + \omega t \cos \omega t$, $f'(0) = 0$, $f'' = 2\omega \cos \omega t - \omega^2 t \sin \omega t$. Hence by (2),

$$\mathcal{L}(f'') = 2\omega \frac{s}{s^2 + \omega^2} - \omega^2\mathcal{L}(f) = s^2\mathcal{L}(f), \qquad \text{thus} \qquad \mathcal{L}(f) = \mathcal{L}(t \sin \omega t) = \frac{2\omega s}{(s^2 + \omega^2)^2}. \qquad \blacksquare$$

EXAMPLE 2 **Formulas 7 and 8 in Table 6.1, Sec. 6.1**

This is a third derivation of $\mathcal{L}(\cos \omega t)$ and $\mathcal{L}(\sin \omega t)$; cf. Example 4 in Sec. 6.1. Let $f(t) = \cos \omega t$. Then $f(0) = 1$, $f'(0) = 0$, $f''(t) = -\omega^2 \cos \omega t$. From this and (2) we obtain

$$\mathcal{L}(f'') = s^2\mathcal{L}(f) - s = -\omega^2\mathcal{L}(f). \qquad \text{By algebra,} \qquad \mathcal{L}(\cos \omega t) = \frac{s}{s^2 + \omega^2}.$$

Similarly, let $g = \sin \omega t$. Then $g(0) = 0$, $g' = \omega \cos \omega t$. From this and (1) we obtain

$$\mathcal{L}(g') = s\mathcal{L}(g) = \omega\mathcal{L}(\cos \omega t). \qquad \text{Hence} \qquad \mathcal{L}(\sin \omega t) = \frac{\omega}{s}\mathcal{L}(\cos \omega t) = \frac{\omega}{s^2 + \omega^2}. \qquad \blacksquare$$

Laplace Transform of the Integral of a Function

Differentiation and integration are inverse operations, and so are multiplication and division. Since differentiation of a function $f(t)$ (roughly) corresponds to multiplication of its transform $\mathcal{L}(f)$ by s, we expect integration of $f(t)$ to correspond to division of $\mathcal{L}(f)$ by s:

THEOREM 3 **Laplace Transform of Integral**

Let $F(s)$ denote the transform of a function $f(t)$ which is piecewise continuous for $t \geqq 0$ and satisfies a growth restriction (2), *Sec.* 6.1. *Then, for $s > 0$, $s > k$, and $t > 0$,*

$$\text{(4)} \qquad \mathcal{L}\left\{\int_0^t f(\tau)\, d\tau\right\} = \frac{1}{s}F(s), \qquad \text{thus} \qquad \int_0^t f(\tau)\, d\tau = \mathcal{L}^{-1}\left\{\frac{1}{s}F(s)\right\}.$$

PROOF Denote the integral in (4) by $g(t)$. Since $f(t)$ is piecewise continuous, $g(t)$ is continuous, and (2), Sec. 6.1, gives

$$|g(t)| = \left|\int_0^t f(\tau)\, d\tau\right| \leqq \int_0^t |f(\tau)|\, d\tau \leqq M \int_0^t e^{k\tau}\, d\tau = \frac{M}{k}(e^{kt} - 1) \leqq \frac{M}{k}e^{kt} \qquad (k > 0).$$

This shows that $g(t)$ also satisfies a growth restriction. Also, $g'(t) = f(t)$, except at points at which $f(t)$ is discontinuous. Hence $g'(t)$ is piecewise continuous on each finite interval and, by Theorem 1, since $g(0) = 0$ (the integral from 0 to 0 is zero)

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{g'(t)\} = s\mathcal{L}\{g(t)\} - g(0) = s\mathcal{L}\{g(t)\}.$$

Division by s and interchange of the left and right sides gives the first formula in (4), from which the second follows by taking the inverse transform on both sides. $\blacksquare$

EXAMPLE 3 **Application of Theorem 3: Formulas 19 and 20 in the Table of Sec. 6.9**

Using Theorem 3, find the inverse of $\dfrac{1}{s(s^2 + \omega^2)}$ and $\dfrac{1}{s^2(s^2 + \omega^2)}$.

Solution. From Table 6.1 in Sec. 6.1 and the integration in (4) (second formula with the sides interchanged) we obtain

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2 + \omega^2}\right\} = \frac{\sin \omega t}{\omega}, \qquad \mathcal{L}^{-1}\left\{\frac{1}{s(s^2 + \omega^2)}\right\} = \int_0^t \frac{\sin \omega\tau}{\omega}\, d\tau = \frac{1}{\omega^2}(1 - \cos \omega t).$$

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This is a third derivation of $\mathscr{L}(\cos \omega t)$ and $\mathscr{L}(\sin \omega t)$; cf. Example 4 in Sec. 6.1. Let $f(t) = \cos \omega t$. Then $f(0) = 1$, $f'(0) = 0$, $f''(t) = -\omega^2 \cos \omega t$. From this and (2) we obtain

$$\mathscr{L}(f'') = s^2\mathscr{L}(f) - s = -\omega^2\mathscr{L}(f). \qquad \text{By algebra,} \qquad \mathscr{L}(\cos \omega t) = \frac{s}{s^2 + \omega^2}.$$

Similarly, let $g = \sin \omega t$. Then $g(0) = 0$, $g' = \omega \cos \omega t$. From this and (1) we obtain

$$\mathscr{L}(g') = s\mathscr{L}(g) = \omega\mathscr{L}(\cos \omega t). \qquad \text{Hence} \qquad \mathscr{L}(\sin \omega t) = \frac{\omega}{s}\mathscr{L}(\cos \omega t) = \frac{\omega}{s^2 + \omega^2}. \qquad \blacksquare$$

Laplace Transform of the Integral of a Function

Differentiation and integration are inverse operations, and so are multiplication and division. Since differentiation of a function $f(t)$ (roughly) corresponds to multiplication of its transform $\mathscr{L}(f)$ by s, we expect integration of $f(t)$ to correspond to division of $\mathscr{L}(f)$ by s:

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$$\text{(4)} \qquad \mathscr{L}\left\{\int_0^t f(\tau)\, d\tau\right\} = \frac{1}{s}F(s), \qquad \text{thus} \qquad \int_0^t f(\tau)\, d\tau = \mathscr{L}^{-1}\left\{\frac{1}{s}F(s)\right\}.$$

PROOF Denote the integral in (4) by $g(t)$. Since $f(t)$ is piecewise continuous, $g(t)$ is continuous, and (2), Sec. 6.1, gives

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$$\mathscr{L}\{f(t)\} = \mathscr{L}\{g'(t)\} = s\mathscr{L}\{g(t)\} - g(0) = s\mathscr{L}\{g(t)\}.$$

Division by s and interchange of the left and right sides gives the first formula in (4), from which the second follows by taking the inverse transform on both sides. $\blacksquare$

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Solution. From Table 6.1 in Sec. 6.1 and the integration in (4) (second formula with the sides interchanged) we obtain

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2 + \omega^2}\right\} = \frac{\sin \omega t}{\omega}, \qquad \mathscr{L}^{-1}\left\{\frac{1}{s(s^2 + \omega^2)}\right\} = \int_0^t \frac{\sin \omega\tau}{\omega}\, d\tau = \frac{1}{\omega^2}(1 - \cos \omega t).$$

EXAMPLE 4 **Initial Value Problem: The Basic Laplace Steps**

Solve

$$y'' - y = t, \qquad y(0) = 1, \qquad y'(0) = 1.$$

Solution. *Step 1.* From (2) and Table 6.1 we get the subsidiary equation [with $Y = \mathcal{L}(y)$]

$$s^2Y - sy(0) - y'(0) - Y = 1/s^2, \qquad \text{thus} \qquad (s^2 - 1)Y = s + 1 + 1/s^2.$$

Step 2. The transfer function is $Q = 1/(s^2 - 1)$, and (7) becomes

$$Y = (s + 1)Q + \frac{1}{s^2}\,Q = \frac{s+1}{s^2 - 1} + \frac{1}{s^2(s^2 - 1)}\,.$$

Simplification and **partial fraction expansion** gives

$$Y = \frac{1}{s - 1} + \left(\frac{1}{s^2 - 1} - \frac{1}{s^2}\right).$$

Step 3. From this expression for Y and Table 6.1 we obtain the solution

$$y(t) = \mathcal{L}^{-1}(Y) = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2 - 1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = e^t + \sinh t - t.$$

The diagram in Fig. 115 summarizes our approach. ■

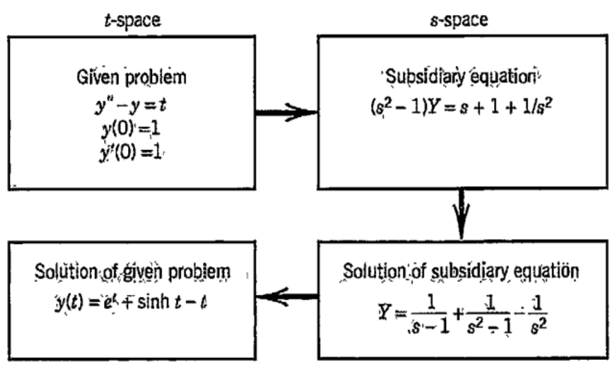

Fig. 115. Laplace transform method

EXAMPLE 5 **Comparison with the Usual Method**

Solve the initial value problem

$$y'' + y' + 9y = 0, \qquad y(0) = 0.16, \qquad y'(0) = 0.$$

Solution. From (1) and (2) we see that the subsidiary equation is

$$s^2Y - 0.16s + sY - 0.16 + 9Y = 0, \qquad \text{thus} \qquad (s^2 + s + 9)Y = 0.16(s + 1).$$

The solution is

$$Y = \frac{0.16(s+1)}{s^2 + s + 9} = \frac{0.16(s + \frac{1}{2}) + 0.08}{(s + \frac{1}{2})^2 + \frac{35}{4}}\,.$$

Hence by the first shifting theorem and the formulas for cos and sin in Table 6.1 we obtain

$$y(t) = \mathcal{L}^{-1}(Y) = e^{-t/2}\left(0.16\cos\sqrt{\frac{35}{4}}\,t + \frac{0.08}{\frac{1}{2}\sqrt{35}}\sin\sqrt{\frac{35}{4}}\,t\right)$$

$$= e^{-0.5t}(0.16\cos 2.96t + 0.027\sin 2.96t).$$

This agrees with Example 2, Case (III) in Sec. 2.4. The work was less. ■

Advantages of the Laplace Method

1. *Solving a nonhomogeneous ODE does not require first solving the homogeneous ODE.* See Example 4.
2. *Initial values are automatically taken care of.* See Examples 4 and 5.
3. *Complicated inputs* $r(t)$ (right sides of linear ODEs) *can be handled very efficiently,* as we show in the next sections.

EXAMPLE 6 Shifted Data Problems

This means initial value problems with initial conditions given at some $t = t_0 > 0$ instead of $t = 0$. For such a problem set $t = \tilde{t} + t_0$, so that $t = t_0$ gives $\tilde{t} = 0$ and the Laplace transform can be applied. For instance, solve

$$y'' + y = 2t, \qquad y(\tfrac{1}{4}\pi) = \tfrac{1}{2}\pi, \qquad y'(\tfrac{1}{4}\pi) = 2 - \sqrt{2}.$$

Solution. We have $t_0 = \frac{1}{4}\pi$ and we set $t = \tilde{t} + \frac{1}{4}\pi$. Then the problem is

$$\tilde{y}'' + \tilde{y} = 2(\tilde{t} + \tfrac{1}{4}\pi), \qquad \tilde{y}(0) = \tfrac{1}{2}\pi, \qquad \tilde{y}'(0) = 2 - \sqrt{2}$$

where $\tilde{y}(\tilde{t}) = y(t)$. Using (2) and Table 6.1 and denoting the transform of $\tilde{y}$ by $\tilde{Y}$, we see that the subsidiary equation of the "shifted" initial value problem is

$$s^2\tilde{Y} - s\cdot\tfrac{1}{2}\pi - (2 - \sqrt{2}) + \tilde{Y} = \frac{2}{s^2} + \frac{\frac{1}{2}\pi}{s}, \qquad \text{thus} \qquad (s^2 + 1)\tilde{Y} = \frac{2}{s^2} + \frac{\frac{1}{2}\pi}{s} + \frac{1}{2}\pi s + 2 - \sqrt{2}.$$

Solving this algebraically for $\tilde{Y}$, we obtain

$$\tilde{Y} = \frac{2}{(s^2 + 1)s^2} + \frac{\frac{1}{2}\pi}{(s^2 + 1)s} + \frac{\frac{1}{2}\pi s}{s^2 + 1} + \frac{2 - \sqrt{2}}{s^2 + 1}.$$

The inverse of the first two terms can be seen from Example 3 (with $\omega = 1$), and the last two terms give cos and sin,

$$\begin{aligned}\tilde{y} = \mathcal{L}^{-1}(\tilde{Y}) &= 2(\tilde{t} - \sin\tilde{t}) + \tfrac{1}{2}\pi(1 - \cos\tilde{t}) + \tfrac{1}{2}\pi\cos\tilde{t} + (2 - \sqrt{2})\sin\tilde{t} \\ &= 2\tilde{t} + \tfrac{1}{2}\pi - \sqrt{2}\sin\tilde{t}.\end{aligned}$$

Now $\tilde{t} = t - \frac{1}{4}\pi$, $\sin\tilde{t} = \dfrac{1}{\sqrt{2}}(\sin t - \cos t)$, so that the answer (the solution) is

$$y = 2t - \sin t + \cos t.$$

■

PROBLEM SET 6.2

1–8 OBTAINING TRANSFORMS BY DIFFERENTIATION

Using (1) or (2), find $\mathcal{L}(f)$ if $f(t)$ equals:

1. te^{kt}
2. $t \cos 5t$
3. $\sin^2 \omega t$
4. $\cos^2 \pi t$
5. $\sinh^2 at$
6. $\cosh^2 \frac{1}{2}t$
7. $t \sin \frac{1}{2}\pi t$
8. $\sin^4 t$ (Use Prob. 3.)
9. **(Derivation by different methods)** It is typical that various transforms can be obtained by several methods. Show this for Prob. 1. Show it for $\mathcal{L}(\cos^2 \frac{1}{2}t)$ (a) by expressing $\cos^2 \frac{1}{2}t$ in terms of $\cos t$, (b) by using Prob. 3.

10–24 INITIAL VALUE PROBLEMS

Solve the following initial value problems by the Laplace transform. (If necessary, use partial fraction expansion as in Example 4. Show all details.)

10. $y' + 4y = 0, \quad y(0) = 2.8$
11. $y' + \frac{1}{2}y = 17 \sin 2t, \quad y(0) = -1$
12. $y'' - y' - 6y = 0, \quad y(0) = 6, \quad y'(0) = 13$
13. $y'' - \frac{1}{4}y = 0, \quad y(0) = 4, \quad y'(0) = 0$
14. $y'' - 4y' + 4y = 0, \quad y(0) = 2.1, \quad y'(0) = 3.9$
15. $y'' + 2y' + 2y = 0, \quad y(0) = 1, \quad y'(0) = -3$
16. $y'' + ky' - 2k^2y = 0, \quad y(0) = 2, \quad y'(0) = 2k$
17. $y'' + 7y' + 12y = 21e^{3t}, \quad y(0) = 3.5, \quad y'(0) = -10$
18. $y'' + 9y = 10e^{-t}, \quad y(0) = 0, \quad y'(0) = 0$
19. $y'' + 3y' + 2.25y = 9t^3 + 64, \quad y(0) = 1, \quad y'(0) = 31.5$
20. $y'' - 6y' + 5y = 29 \cos 2t, \quad y(0) = 3.2, \quad y'(0) = 6.2$
21. **(Shifted data)** $y' - 6y = 0, \quad y(2) = 4$
22. $y'' - 2y' - 3y = 0, \quad y(1) = -3, \quad y'(1) = -17$
23. $y'' + 3y' - 4y = 6e^{2t-2}, \quad y(1) = 4, \quad y'(1) = 5$
24. $y'' + 2y' + 5y = 50t - 150, \quad y(3) = -4, \quad y'(3) = 14$
25. **PROJECT. Comments on Sec. 6.2. (a)** Give reasons why Theorems 1 and 2 are more important than Theorem 3.

    (b) Extend Theorem 1 by showing that if $f(t)$ is continuous, except for an ordinary discontinuity (finite jump) at some $t = a$ (> 0), the other conditions remaining as in Theorem 1, then (see Fig. 116)

$$(1^*) \quad \mathcal{L}(f') = s\mathcal{L}(f) - f(0) - [f(a + 0) - f(a - 0)]e^{-as}.$$

    (c) Verify (1*) for $f(t) = e^{-t}$ if $0 < t < 1$ and 0 if $t > 1$.

    (d) Verify (1*) for two more complicated functions of your choice.

    (e) Compare the Laplace transform of solving ODEs with the method in Chap. 2. Give examples of your own to illustrate the advantages of the present method (to the extent we have seen them so far).

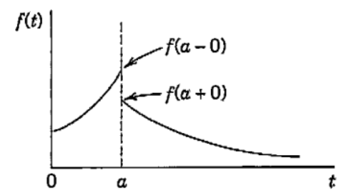

Fig. 116. Formula (1*)

26. **PROJECT. Further Results by Differentiation.** Proceeding as in Example 1, obtain

$$\text{(a)} \quad \mathcal{L}(t \cos \omega t) = \frac{s^2 - \omega^2}{(s^2 + \omega^2)^2}$$

    and from this and Example 1: (b) formula 21, (c) 22, (d) 23 in Sec. 6.9,

$$\text{(e)} \quad \mathcal{L}(t \cosh at) = \frac{s^2 + a^2}{(s^2 - a^2)^2},$$

$$\text{(f)} \quad \mathcal{L}(t \sinh at) = \frac{2as}{(s^2 - a^2)^2}.$$

27–34 OBTAINING TRANSFORMS BY INTEGRATION

Using Theorem 3, find $f(t)$ if $\mathcal{L}(f)$ equals:

27. $\dfrac{1}{s^2 + s/2}$
28. $\dfrac{10}{s^3 - \pi s^2}$
29. $\dfrac{1}{s^3 - ks^2}$
30. $\dfrac{1}{s^4 + s^2}$
31. $\dfrac{5}{s^3 - 5s}$
32. $\dfrac{2}{s^3 + 9s}$
33. $\dfrac{1}{s^4 - 4s^2}$
34. $\dfrac{1}{s^4 + \pi^2 s^2}$
35. **(Partial fractions)** Solve Probs. 27, 29, and 31 by using partial fractions.