

### 3.rd week

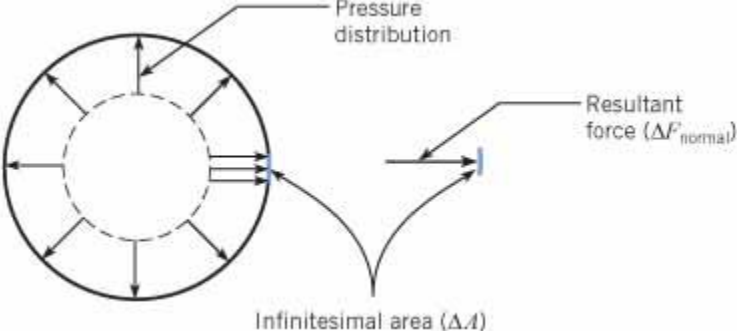
## Fluid Static

### Definitions, Pressure at a point, Variation of Pressure in a static fluid

Fluid Statics When the fluid velocity is zero, called the hydrostatic condition.

#### Definition of Pressure

*Pressure* is defined as the ratio of normal force to area at a point. For example, Fig. 3.2 shows fluid inside an object such as air inside a soccer ball. The molecules of the fluid interact with the walls to produce a pressure distribution. At each point on the walls, this pressure distribution creates a resultant force  $F$  normal that acts on an infinitesimal unit of area  $A$  as shown. Pressure is the ratio of normal force magnitude  $|\Delta \vec{F}_{\text{normal}}|$  to unit area  $A$  at a point:

$$p = \lim_{\Delta A \rightarrow 0} \frac{|\Delta \vec{F}_{\text{normal}}|}{\Delta A} = \frac{dF_{\text{normal}}}{dA}$$


The diagram shows a sphere with a dashed inner circle. Arrows of varying lengths point radially outward from the sphere's surface, representing a pressure distribution. A specific point on the surface is highlighted with a small blue rectangle, labeled 'Infinitesimal area ( $\Delta A$ )'. A curved arrow points from this area to a larger blue rectangle, labeled 'Resultant force ( $\Delta F_{\text{normal}}$ )'. A label 'Pressure distribution' points to the arrows on the sphere's surface.

**Figure** *Pressure acting on the walls of a sphere*

The reason that pressure is defined using a derivative is that pressure often varies from point to point. For example, pressure acting on the windshield of a moving car will vary at different locations on the windshield.

Pressure is a scalar quantity; that is, it has magnitude only. Pressure is not a force; rather it is a scalar that produces a resultant force by its action on an area. The resultant force is normal to the area and acts in a direction toward the surface (compressive).

The pressure variation is due only to the weight of the fluid. Consider a small wedge of fluid at rest of size  $\Delta x$ ,  $\Delta z$ ,  $\Delta s$  and depth  $b$  into the paper.

There is no shear stress by definition, and pressure is assumed to be identical on each face (small element).

Since the element is at rest, summation of all forces must equal zero.

$$\sum F_x = 0 = P_x b \Delta z - P_n b \Delta s \sin \theta$$

$$\sum F_z = 0 = P_z b \Delta x - P_n b \Delta s \cos \theta - \frac{1}{2} \rho g b \Delta x \Delta z$$

From geometry:  $\Delta s \sin \theta = \Delta z$ ,  $\Delta s \cos \theta = \Delta x$

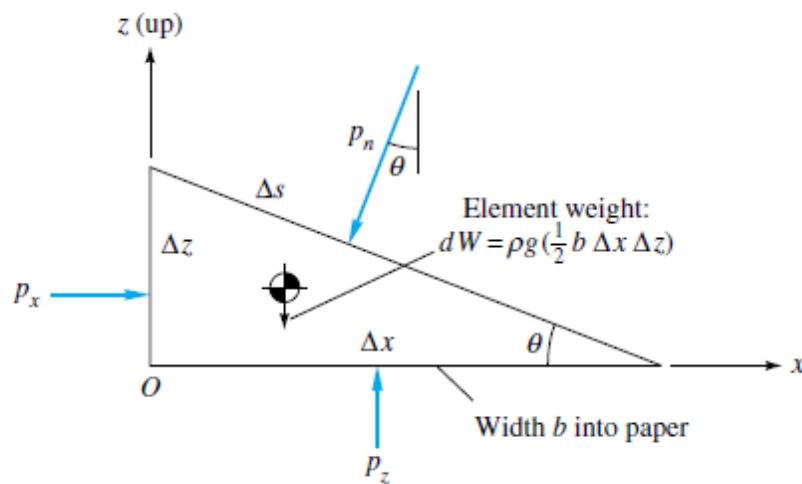


Fig. 1: Equilibrium of a small Fluid element at rest.

After substitution in above equations, one finds:

$$P_x = P_n. P_z = P_n + \frac{1}{2} \rho g \Delta z$$

This means: 1) There is no pressure change in the horizontal direction. 2) There is a vertical change in pressure proportional to the density, gravity and depth change in the fluid (i.e. the weight of the column of the fluid above the point). Note: in the limit as the fluid wedge shrinks to a point,  $\Delta z$  goes to zero, we have:

$P_x = P_z = P_n = P$  Thus, pressure in a static fluid is a point property.

Pressure force on a fluid element

Assume the pressure vary arbitrarily in a fluid,  $p=p(x,y,z,t)$ . Consider a fluid element of size  $\Delta x$ ,  $\Delta y$ ,  $\Delta z$  as shown in Fig. 2. The net force in the x-direction is given by:

$$dF_x = P dy dz - \left( P + \frac{\partial P}{\partial x} dx \right) dy dz = -\frac{\partial P}{\partial x} dx dy dz$$

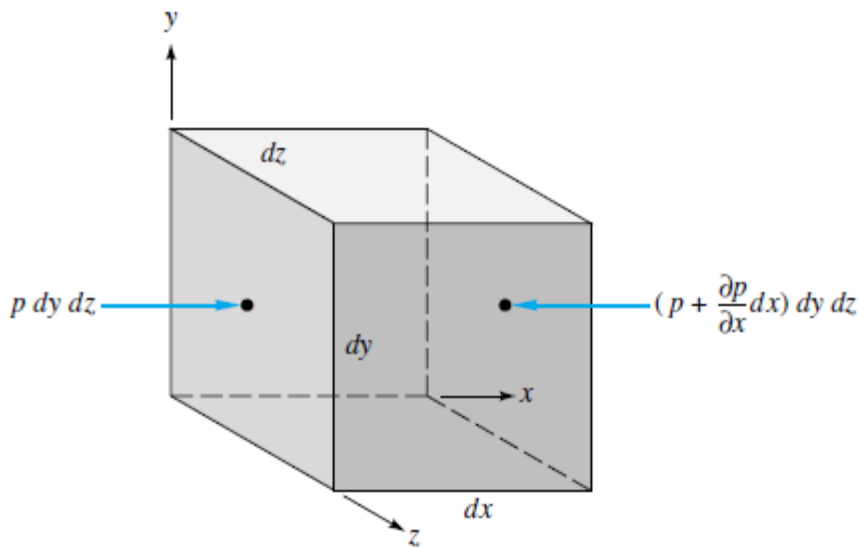


Fig. 2: Net force in the x-direction due to pressure variation

In a similar manner, net forces acting in y- and z-directions can be calculated. The total net force vector, due to pressure, is:

$$dF_{press} = -\left( \frac{\partial P}{\partial x} i + \frac{\partial P}{\partial y} j + \frac{\partial P}{\partial z} k \right) dx dy dz$$

Notice that the term in the parentheses is the negative vector gradient of pressure and the term  $dx \, dy \, dz = dV$ , is the volume of the element.

Therefore, one can write:

$$f_{press} = -\nabla P$$

where  $f_{press}$  is the net force per volume. Notice that the pressure gradient (not pressure) causing a net force that must be balanced by gravity or acceleration and/or other effects in the fluid. Note: the pressure gradient is a surface force that acts on the sides of the element. That must be

balanced by gravity force, or weight of the element, in the fluid at rest. In addition to gravity, a fluid in motion will have surface forces due to viscous stresses. Viscous forces, however, for a fluid at rest are zero.

## 1.7 PRESSURE AND A PERFECT GAS

As will be elaborated upon in Chap. 2 the *pressure* at a point derives from a normal force pushing against a plane defined in the fluid or a plane surface that the fluid is in contact with. The pressure at a point is the ratio of the normal force to the area of the plane as the area approaches a small value enclosing the point. Pressure,  $p$ , has units of force per unit area, which may be newtons per square meter, called pascals (Pa), or pounds per square foot (psf), or pounds per square inch (psi).

Liquids normally cannot sustain a tensile (or pulling apart) stress since the liquid would vaporize. Therefore, the absolute pressures used in this book are never

negative, since this would imply that the fluid is sustaining a tensile stress. Liquids can often sustain a considerable pressure or compressive force with little or no change in the observed density. However, no universal relationships between pressure and density exist for a liquid.

Gas responds to changes in pressure or compressive force. For ideal gases a relationship between pressure and density can be explicitly stated. The perfect gas, as used herein, is defined as a substance that satisfies the *perfect-gas law*

$$pv_s = RT \quad (1.7.1)$$

and that has constant specific heats;  $p$  is the absolute pressure,  $v_s$  the specific volume,  $R$  the gas constant, and  $T$  the absolute temperature. The perfect gas must be carefully distinguished from the ideal fluid. An ideal fluid is frictionless and incompressible. The perfect gas has viscosity and can therefore develop shear stresses, and it is compressible according to Eq. (1.7.1).

Equation (1.7.1) is the equation of state for a perfect gas. It can be written as

$$p = \rho RT \quad (1.7.2)$$

The units of  $R$  can be determined from the equation when the other units are known. For  $p$  in pascals,  $\rho$  in kilograms per cubic meter, and  $T$  in kelvins (K)

$$R = \frac{\text{N}}{\text{m}^2} \frac{\text{m}^3}{\text{kg} \cdot \text{K}} = \frac{\text{m} \cdot \text{N}}{\text{kg} \cdot \text{K}} \quad \text{or} \quad \text{m} \cdot \text{N} / \text{kg} \cdot \text{K}$$

For USC units,  $^{\circ}\text{R} = ^{\circ}\text{F} + 459.6$

$$R = \frac{\text{lb}}{\text{ft}^2} \frac{\text{ft}^3}{\text{slug} \cdot ^{\circ}\text{R}} = \frac{\text{ft} \cdot \text{lb}}{\text{slug} \cdot ^{\circ}\text{R}} \quad \text{or} \quad \text{ft} \cdot \text{lb} / \text{slug} \cdot ^{\circ}\text{R}$$

For  $\rho$  in pounds mass per cubic foot

$$R = \frac{\text{lb}}{\text{ft}^2} \frac{\text{ft}^3}{\text{lb}_m \cdot ^{\circ}\text{R}} = \frac{\text{ft} \cdot \text{lb}}{\text{lb}_m \cdot ^{\circ}\text{R}} \quad \text{or} \quad \text{ft} \cdot \text{lb} / \text{lb}_m \cdot ^{\circ}\text{R}$$

The magnitude of  $R$  in slugs is 32.174 times greater than in pounds mass. Values of  $R$  for several common gases are given in Table C.3 of Appendix C.

Real gases below critical pressure and above the critical temperature tend to obey the perfect-gas law. As the pressure increases, the discrepancy increases and becomes serious near the critical point. The perfect-gas law encompasses both Charles' law and Boyle's law. Charles' law states that for constant pressure the volume of a given mass of gas varies as its absolute temperature. Boyle's law (isothermal law) states that for constant temperature the density varies directly as the absolute pressure. The volume  $V$  of  $m$  mass units of gas is  $mv_s$ ; hence,

$$pV = mRT \quad (1.7.3)$$

Certain simplifications result from writing the perfect-gas law on a mole basis. A kilogram mole of gas is the number of kilograms mass of gas equal to the molecular weight; for example, a kilogram mole of oxygen  $O_2$  is 32 kg. With  $\bar{v}_s$  being the volume per mole, the perfect-gas law becomes

$$p\bar{v}_s = MRT \quad (1.7.4)$$

if  $M$  is the molecular weight. In general, if  $n$  is the number of moles of the gas in volume  $V$ , then

$$pV = nMRT \quad (1.7.5)$$

since  $nM = m$ . Now, from Avogadro's law, equal volumes of gases at the same absolute temperature and pressure have the same number of molecules; hence their masses are proportional to the molecular weights. From Eq. (1.7.5) it is seen that  $MR$  must be constant, since  $pV/nT$  is the same for any perfect gas. The product  $MR$ , called the *universal gas constant*, has a value depending only upon the units employed. It is

$$MR = 8312 \text{ m}\cdot\text{N/kg}\cdot\text{mol}\cdot\text{K} \quad (1.7.6)$$

The gas constant  $R$  can then be determined from

$$R = \frac{8312}{M} \text{ m}\cdot\text{N/kg}\cdot\text{K} \quad (1.7.7)$$

In USC units

$$R = \frac{49,709}{M} \text{ ft}\cdot\text{lb/slug}\cdot^\circ\text{R} \quad (1.7.8)$$

In pounds mass units

$$R = \frac{1545}{M} \text{ ft}\cdot\text{lb/lb}_m\cdot^\circ\text{R} \quad (1.7.9)$$

# PHYSICAL PROPERTIES OF FLUIDS

TABLE C.1 Physical properties of water in SI units

Temp, °C	Specific weight $\gamma$ , N/m <sup>3</sup>	Density $\rho$ , kg/m <sup>3</sup>	Viscosity $\mu \times 10^3$ , N·s/m <sup>2</sup>	Kine- matic viscosity $\nu \times 10^6$ , m <sup>2</sup> /s	Surface tension $\sigma \times 10^2$ , N/m	Vapor- pressure head absolute $p_v/\gamma$ , <sup>†</sup> m	Bulk modulus of elasticity $K \times 10^{-7}$ , N/m <sup>2</sup>	Thermal conduct- ivity $k$ , W/m·K
0	9806	999.9	1.792	1.792	7.62	0.06	204	0.561
5	9807	1000.0	1.519	1.519	7.54	0.09	206	0.571
10	9804	999.7	1.308	1.308	7.48	0.12	211	0.580
15	9798	999.1	1.140	1.141	7.41	0.17	214	0.589
20	9789	998.2	1.005	1.007	7.36	0.25	220	0.598
25	9778	997.1	0.894	0.897	7.26	0.33	222	0.607
30	9764	995.7	0.801	0.804	7.18	0.44	223	0.615
35	9749	994.1	0.723	0.727	7.10	0.58	224	0.623
40	9730	992.2	0.656	0.661	7.01	0.76	227	0.630
45	9711	990.2	0.599	0.605	6.92	0.98	229	0.637
50	9690	988.1	0.549	0.556	6.82	1.26	230	0.643
55	9666	985.7	0.506	0.513	6.74	1.61	231	0.649
60	9642	983.2	0.469	0.477	6.68	2.03	228	0.654
65	9616	980.6	0.436	0.444	6.58	2.56	226	0.659
70	9589	977.8	0.406	0.415	6.50	3.20	225	0.663
75	9560	974.9	0.380	0.390	6.40	3.96	223	0.667
80	9530	971.8	0.357	0.367	6.30	4.86	221	0.670
85	9499	968.6	0.336	0.347	6.20	5.93	217	0.673
90	9466	965.3	0.317	0.328	6.12	7.18	216	0.675
95	9433	961.9	0.299	0.311	6.02	8.62	211	0.677
100	9399	958.4	0.284	0.296	5.94	10.33	207	0.679

<sup>†</sup>  $\gamma = 9806 \text{ N/m}^3$ .

Table 1.3 Approximate properties of common liquids at 20°C and standard atmospheric pressure

Liquid	Specific gravity $S$	Bulk modulus of elasticity $K$ , GPa	Vapor pressure $p_v$ , kPa	Surface tension <sup>†</sup> $\sigma$ , N/m
Alcohol, ethyl	0.79	1.21	5.86	0.0223
Benzene	0.88	1.03	10.0	0.0289
Carbon tetrachloride	1.59	1.10	13.1	0.0267
Kerosene	0.81	...	...	0.023–0.032
Mercury	13.57	26.20	0.00017	0.51
Oil:				
Crude	0.85–0.93	...	...	0.023–0.038
Lubricating	0.85–0.88	...	...	0.023–0.038
Water	1.00	2.2	2.45	0.074

so that knowledge of molecular weight leads to the value of  $R$ . In Table C.3 of Appendix C molecular weights of some common gases are listed. Additional relations and definitions used in perfect-gas flow are introduced in Chap. 3.

A gas with molecular weight of 44 is at a pressure of 0.9 MPa and a temperature of 20°C. Determine its density.

**Example 1.3**

**Solution**

From Eq. (1.7.7)

$$R = \frac{8312}{44} = 188.91 \text{ m}\cdot\text{N/kg}\cdot\text{K}$$

Then, from Eq. (1.7.2)

$$\rho = \frac{p}{RT} = \frac{0.9 \times 10^6 \text{ N/m}^2}{(188.91 \text{ m}\cdot\text{N/kg}\cdot\text{K})(273 + 20 \text{ K})} = 16.26 \text{ kg/m}^3$$

## EXERCISES

- 1.7.1** A perfect gas (a) has zero viscosity; (b) has constant viscosity; (c) is incompressible; (d) satisfies  $p\rho = RT$ ; (e) fits none of these statements.
- 1.7.2** The molecular weight of a gas is 28. The value of  $R$  in meter-newtons per kilogram-kelvin is (a) 29.7; (b) 297; (c) 2911; (d) 8312; (e) none of these answers.
- 1.7.3** The density of air at 10°C and 1-MPa absolute (abs) in SI units is (a) 1.231; (b) 12.31; (c) 65.0; (d) 118.4; (e) none of these answers.
- 1.7.4** How many kilograms mass of carbon monoxide gas at 20°C and 200-kPa abs is contained in a volume of 100 L? (a) 0.00023; (b) 0.23; (c) 3.367; (d) 3367; (e) none of these answers.



The last term of the second equation is an infinitesimal of higher order of smallness and can be neglected. When divided by  $\delta y$  and  $\delta x$ , respectively, the equations can be combined:

$$p_x = p_y = p_z \quad (2.1.4)$$

Since  $\theta$  is any arbitrary angle, this equation proves that the pressure is the same in all directions at a point in a static fluid. Although the proof was carried out for a two-dimensional case, it can be demonstrated for the three-dimensional case with the equilibrium equations for a small tetrahedron of fluid with three faces in the coordinate planes and the fourth face inclined arbitrarily.

If the fluid is in motion so that one layer moves relative to an adjacent layer, shear stresses occur and the normal stresses are, in general, no longer the same in all directions at a point. The pressure is then defined as the average of any three mutually perpendicular normal compressive stresses at a point [Eq. (2.1.3)]. In a fictitious fluid of zero viscosity, that is, a frictionless fluid, no shear stresses can occur for any motion of the fluid, and so at a point the pressure is the same in all directions.

## EXERCISE

**2.1.1** The normal stress is the same in all directions at a point in a fluid (a) *only* when the fluid is frictionless; (b) *only* when the fluid is frictionless and incompressible; (c) *only* when the fluid has zero viscosity and is at rest; (d) when there is no motion of one fluid layer relative to an adjacent layer; (e) regardless of the motion of one fluid layer relative to an adjacent layer.

## 2.2 BASIC EQUATION OF FLUID STATICS

### Pressure Variation in a Static Fluid

The forces acting on an element of fluid at rest (Fig. 2.5) consist of surface forces and body forces. With gravity the only body force acting, by taking the  $y$  axis vertically upward it is  $-\gamma \delta x \delta y \delta z$  in the  $y$  direction. With pressure  $p$  at its center ( $x, y, z$ ), the force exerted on the side normal to the  $y$  axis closest to the origin is approximately

$$\left( p - \frac{\partial p}{\partial y} \frac{\delta y}{2} \right) \delta x \delta z$$

and the force exerted on the opposite side is

$$\left( p + \frac{\partial p}{\partial y} \frac{\delta y}{2} \right) \delta x \delta z$$

where  $\delta y/2$  is the distance from the center to a face normal to  $y$ . Summing the forces acting on the element in the  $y$  direction gives

$$\delta F_y = -\frac{\partial p}{\partial y} \delta x \delta y \delta z - \gamma \delta x \delta y \delta z$$

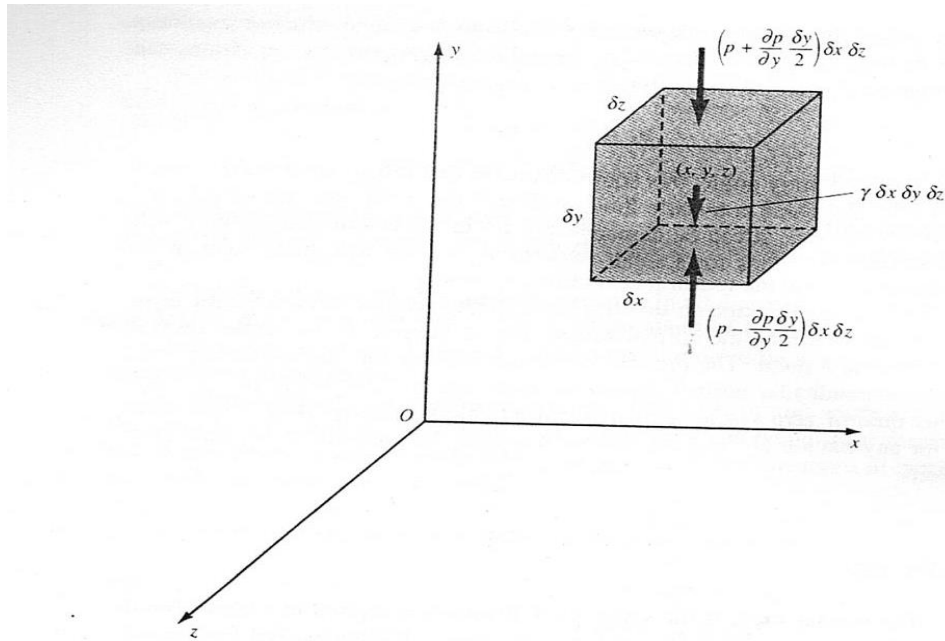


Figure 2.5 Rectangular parallelepiped element of fluid at rest.

For the  $x$  and  $z$  directions, since no body forces act,

$$\delta F_x = -\frac{\partial p}{\partial x} \delta x \delta y \delta z \quad \delta F_z = -\frac{\partial p}{\partial z} \delta x \delta y \delta z$$

The elemental force vector  $\delta \mathbf{F}$  is given by

$$\delta \mathbf{F} = \mathbf{i} \delta F_x + \mathbf{j} \delta F_y + \mathbf{k} \delta F_z = -\left(\mathbf{i} \frac{\partial p}{\partial x} + \mathbf{j} \frac{\partial p}{\partial y} + \mathbf{k} \frac{\partial p}{\partial z}\right) \delta x \delta y \delta z - \mathbf{j} \gamma \delta x \delta y \delta z$$

If the element is reduced to zero size, after dividing through by  $\delta x \delta y \delta z = \delta V$ , the expression becomes exact

$$\frac{\delta \mathbf{F}}{\delta V} = -\left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z}\right) p - \mathbf{j} \gamma \quad \lim \delta V \rightarrow 0 \quad (2.2.1)$$

This is the resultant force per unit volume at a point, which must be equated to zero for a fluid at rest. The quantity in parentheses is the *gradient*, called  $\nabla$  (del),

$$\nabla = \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \quad (2.2.2)$$

and the negative gradient of  $p$ ,  $-\nabla p$ , is the vector field  $\mathbf{f}$  of the surface pressure force per unit volume,

$$\mathbf{f} = -\nabla p \quad (2.2.3)$$

The fluid static law of variation of pressure is then

$$\mathbf{f} - \mathbf{j}\gamma = 0 \quad (2.2.4)$$

For an inviscid fluid in motion or a fluid moving so that the shear stress is everywhere zero, Newton's second law takes the form

$$\mathbf{f} - \mathbf{j}\gamma = \rho\mathbf{a} \quad (2.2.5)$$

where  $\mathbf{a}$  is the acceleration of the fluid element and  $\mathbf{f} - \mathbf{j}\gamma$  is the resultant fluid force when gravity is the only force acting on the body. Equation (2.2.5) is used to study relative equilibrium in Sec. 2.9 and in the derivation of Euler's equations in Chaps. 4 and 7.

In component form, the combination of Eqs. (2.2.3) and (2.2.4) becomes

$$\frac{\partial p}{\partial x} = 0 \quad \frac{\partial p}{\partial y} = -\gamma \quad \frac{\partial p}{\partial z} = 0 \quad (2.2.6)$$

The partial differential equations, for variation in horizontal directions, are one form of Pascal's law; they state that two points at the same elevation in the same continuous mass of fluid at rest have the same pressure.

Since  $p$  is a function of  $y$  only,

$$dp = -\gamma dy \quad (2.2.7)$$

This simple differential equation relates the change of pressure to specific weight and change of elevation and holds for both compressible and incompressible fluids.

### Pressure Variation in an Incompressible Fluid

For fluids that can be considered homogeneous and incompressible,  $\gamma$  is constant and Eq. (2.2.7), when integrated, becomes

$$p = -\gamma y + c$$

in which  $c$  is the constant of integration. The hydrostatic law of variation of pressure is frequently written in the form

$$p = \gamma h \quad (2.2.8)$$

in which  $h$  is measured vertically downward ( $h = -y$ ) from a free-liquid surface and  $p$  is the increase in pressure from that found at the free surface. Equation (2.2.8) can be derived by taking as a fluid-free body a vertical column of liquid of finite height  $h$  with its upper surface in the free surface. This is left as an exercise for the student.

An oceanographer needs to design a sea lab 5 m high that will withstand submersion to 100 m, measured from sea level to the top of the sea lab. Find the pressure variation on a side of the container and the pressure on the top if the specific gravity of salt water is 1.020.

#### Example 2.1

**Solution**

$$\gamma = 1.020(9806 \text{ N/m}^3) = 10 \text{ kN/m}^3$$

At the top  $h = 100 \text{ m}$ , and

$$p = \gamma h = 1 \text{ MN/m}^2 = 1 \text{ MPa}$$

If  $y$  is measured from the top of the sea lab downward, the pressure variation is

$$p = 10(y + 100) \text{ kPa}$$

Often due to differential heating or the presence of added mass such as salt or sediment, the density in a static homogeneous incompressible fluid may *stratify* or arrange itself in layers where heavier, more dense fluid underlays lighter fluid. The density in each layer remains constant and the pressure varies linearly or hydrostatically with increasing depth into the water column. Figure 2.6 contains an idealized density versus depth diagram of a body of salt water with three constant density regions. The figure also contains the plot of the pressure distribution with depth and it is noted that the pressure is continuous at the interfaces. In practice molecular and turbulent diffusion of salt will marginally “smooth” the discontinuous density interface, but this layered approach to pressure distributions in layered stratified conditions has been a fundamental analysis approach for limnologists and oceanographers for over a hundred years.

From Fig. 2.6 several features are of note. First the pressure in each layer linearly increases with increasing depth. Therefore within layer 1 and subsequent layers the pressure variations are

$$p(0 < h < h_1) = p_0 + \rho_1 g(h) \quad (2.2.9)$$

$$p(h_1 < h < h_2) = p_1 + \rho_2 g(h - h_1) \quad (2.2.10)$$

or for any layer  $n$ ,

$$p(h_{n-1} < h < h_n) = p_{n-1} + \rho_n g(h - h_{n-1}) \quad (2.2.11)$$

### Example 2.2

At a spot in the ocean where the total depth is 450 m, oceanographers measure data at  $h_1 = 100$  m,  $h_2 = 300$  m, and  $h_3 = 450$  m. The specific gravity values for salt

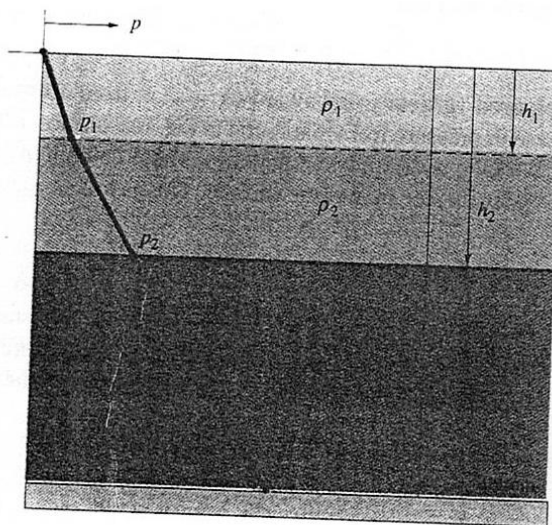


Figure 2.6 Pressure distribution in density stratified fluid at rest.

water in each of the constant density layers are 1.01, 1.02, and 1.025, respectively. Find the pressures at the interfaces. Assume atmospheric pressure at the surface,  $p_0 = 0$ .

**Solution**

$$\begin{aligned} p_1 &= p_0 + \rho_1 g(h_1 - 0) = p_0 + S_1 \rho g h_1 \\ &= 0 + 9.904(100) = 990.4 \text{ kPa} \\ p_2 &= p_1 + \rho_2 g(h_2 - h_1) = p_1 + S_2 \rho g(h_2 - h_1) \\ &= 990.4 + 10.0(200) = 2990.8 \text{ kPa} \\ p_3 &= p_2 + \rho_3 g(h_3 - h_2) = p_2 + S_3 \rho g(h_3 - h_2) \\ &= 2990.8 + 10.05(150) = 4498.5 \text{ kPa} \end{aligned}$$

## Pressure Variation in a Compressible Fluid

When the fluid is a perfect gas at rest at constant temperature, from Eq. (1.7.2)

$$\frac{p}{\rho} = \frac{p_0}{\rho_0} \quad (2.2.12)$$

in which  $p$  is the absolute pressure. When the value of  $\gamma$  in Eq. (2.2.7) is replaced by  $\rho g$  and  $\rho$  is eliminated between Eqs. (2.2.7) and (2.2.9),

$$dy = -\frac{p_0}{g\rho_0} \frac{dp}{p} \quad (2.2.13)$$

It must be remembered that if  $\rho$  is in pounds mass per cubic foot, then  $\gamma = g\rho/g_0$  with  $g_0 = 32.174 \text{ lb}_m \cdot \text{ft}/\text{lb} \cdot \text{s}^2$ . If  $p = p_0$  when  $y = y_0$ , integration between limits

$$\int_{y_0}^y dy = -\frac{p_0}{g\rho_0} \int_{p_0}^p \frac{dp}{p}$$

yields

$$y - y_0 = -\frac{p_0}{g\rho_0} \ln \frac{p}{p_0} \quad (2.2.14)$$

in which  $\ln$  is the natural logarithm. Then

$$p = p_0 \exp\left(-\frac{y - y_0}{p_0/g\rho_0}\right) \quad (2.2.15)$$

which is the equation for variation of pressure with elevation in an isothermal gas.

The atmosphere frequently is assumed to have a constant temperature gradient which is expressed by

$$T = T_0 + \beta y \quad (2.2.16)$$

For the standard atmosphere,  $\beta = -0.00357$  degree Fahrenheit per foot ( $-0.00651 \text{ K/m}$ ) up to the stratosphere. The density can be expressed in terms of pressure and elevation from the perfect-gas law:

$$\rho = \frac{p}{RT} = \frac{p}{R(T_0 + \beta y)} \quad (2.2.17)$$

Substitution into  $dp = -\rho g dy$  [Eq. (2.2.7)] permits the variables to be separated and  $p$  to be found in terms of  $y$  by integration.

### Example 2.3

Assuming isothermal conditions to prevail in the atmosphere, compute the pressure and density at 2000-m elevation if  $p = 10^5 \text{ Pa abs}$  and  $\rho = 1.24 \text{ kg/m}^3$  at sea level.

**Solution**

From Eq. (2.2.15)

$$\begin{aligned} p &= (10^5 \text{ N/m}^2) \exp\left\{-\frac{2000 \text{ m}}{(10^5 \text{ N/m}^2)/[(9.806 \text{ m/s}^2)(1.24 \text{ kg/m}^3)]}\right\} \\ &= 78.4 \text{ kPa abs} \end{aligned}$$

Then, from Eq. (2.2.12),

$$\rho = \frac{p_0}{p} \rho_0 = (1.24 \text{ kg/m}^3) \frac{78,400}{100,000} = 0.972 \text{ kg/m}^3$$