it suffices to show that $f_X(x)$ and $f_Y(y)$ are the marginals of $f_{X,Y}(x, y; \alpha)$.

$$\int_{-\infty}^{\infty} f_{X,Y}(x, y; \alpha)\, dy$$

$$= \int_{-\infty}^{\infty} f_X(x) f_Y(y)\{1 + \alpha[2F_X(x) - 1][2F_Y(y) - 1]\}\, dy$$

$$= f_X(x) \int_{-\infty}^{\infty} f_Y(y)\, dy + \alpha f_X(x)[2F_X(x) - 1] \int_{-\infty}^{\infty} [2F_Y(y) - 1] f_Y(y)\, dy$$

$= f_X(x)$, noting that $\int_{-\infty}^{\infty} [2F_Y(y) - 1] f_Y(y)\, dy$

$$= \int_0^1 (2u - 1)\, du = 0$$

by making the transformation $u = F_Y(y)$. ////

3 CONDITIONAL DISTRIBUTIONS AND STOCHASTIC INDEPENDENCE

In the preceding section we defined the joint distribution and joint density functions of several random variables; in this section we define conditional distributions and the related concept of stochastic independence. Most definitions will be given first for only two random variables and later extended to k random variables.

3.1 Conditional Distribution Functions for Discrete Random Variables

Definition 10 **Conditional discrete density function** Let X and Y be jointly discrete random variables with joint discrete density function $f_{X,Y}(\cdot, \cdot)$. The *conditional discrete density function* of Y given $X = x$, denoted by $f_{Y|X}(\cdot|x)$, is defined to be

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x, y)}{f_X(x)}, \tag{5}$$

if $f_X(x) > 0$, where $f_X(x)$ is the marginal density of X evaluated at x. $f_{Y|X}(\cdot|x)$ is undefined for $f_X(x) = 0$. Similarly,

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x, y)}{f_Y(y)}, \tag{6}$$

if $f_Y(y) > 0$. ////

Definition 11 Conditional discrete cumulative distribution If X and Y are jointly discrete random variables, the *conditional cumulative distribution* of Y given $X = x$, denoted by $F_{Y|X}(\cdot \mid x)$, is defined to be $F_{Y|X}(y \mid x) = P[Y \le y \mid X = x]$ for $f_X(x) > 0$. ////

Remark $F_{Y|X}(y \mid x) = \sum_{\{j:\, y_j \le y\}} f_{Y|X}(y_j \mid x)$. ////

EXAMPLE 9 Return to the experiment of tossing two tetrahedra. Let X denote the number on the downturned face of the first and Y the larger of the downturned numbers. What is the density of Y given that $X = 2$?

$$f_{Y|X}(2 \mid 2) = \frac{f_{X,Y}(2,2)}{f_X(2)} = \frac{\frac{2}{16}}{\frac{4}{16}} = \frac{1}{2}$$

$$f_{Y|X}(3 \mid 2) = \frac{f_{X,Y}(2,3)}{f_X(2)} = \frac{\frac{1}{16}}{\frac{4}{16}} = \frac{1}{4}$$

$$f_{Y|X}(4 \mid 2) = \frac{f_{X,Y}(2,4)}{f_X(2)} = \frac{\frac{1}{16}}{\frac{4}{16}} = \frac{1}{4}.$$

Also,

$$f_{Y|X}(y \mid 3) = \begin{cases} \frac{3}{4} & \text{for } y = 3 \\ \frac{1}{4} & \text{for } y = 4. \end{cases}$$ ////

Definition 12 Conditional discrete density function Let $(X_1, \ldots, X_k)$ be a k-dimensional discrete random variable, and let $X_{i_1}, \ldots, X_{i_r}$ and $X_{j_1}, \ldots, X_{j_s}$ be two disjoint subsets of the random variables $X_1, \ldots, X_k$. The *conditional density* of the r-dimensional random variable $(X_{i_1}, \ldots, X_{i_r})$ given the value $(x_{j_1}, \ldots, x_{j_s})$ of $(X_{j_1}, \ldots, X_{j_s})$ is defined to be

$$f_{X_{i_1}, \ldots, X_{i_r} | X_{j_1}, \ldots, X_{j_s}}(x_{i_1}, \ldots, x_{i_r} \mid x_{j_1}, \ldots, x_{j_s}) = \frac{f_{X_{i_1}, \ldots, X_{i_r}, X_{j_1}, \ldots, X_{j_s}}(x_{i_1}, \ldots, x_{i_r}, x_{j_1}, \ldots, x_{j_s})}{f_{X_{j_1}, \ldots, X_{j_s}}(x_{j_1}, \ldots, x_{j_s})}.$$ ////

EXAMPLE 10 Let $X_1, \ldots, X_5$ be jointly discrete random variables. Take $r = s = 2$, $(X_{i_1}, X_{i_2}) = (X_1, X_2)$, and $(X_{j_1}, X_{j_2}) = (X_3, X_5)$; then

$$f_{X_1, X_2 | X_3, X_5}(x_1, x_2 \mid x_3, x_5) = \frac{f_{X_1, X_2, X_3, X_5}(x_1, x_2, x_3, x_5)}{f_{X_3, X_5}(x_3, x_5)}.$$ ////

EXAMPLE 11 Suppose 12 cards are drawn without replacement from an ordinary deck of playing cards. Let X_1 be the number of aces drawn, X_2 be the number of 2s, X_3 be the number of 3s, and X_4 be the number of 4s. The joint density of these four random variables is given by

$$f_{X_1, X_2, X_3, X_4}(x_1, x_2, x_3, x_4) = \frac{\binom{4}{x_1}\binom{4}{x_2}\binom{4}{x_3}\binom{4}{x_4}\binom{36}{12 - x_1 - x_2 - x_3 - x_4}}{\binom{52}{12}},$$

where $x_i = 0, 1, 2, 3,$ or 4 and $i = 1, \ldots, 4$, subject to the restriction that $\sum x_i \le 12$. There are a large number of conditional densities associated with this density; an example is

$$f_{X_2, X_4 | X_1, X_3}(x_2, x_4 | x_1, x_3) = \frac{\binom{4}{x_1}\binom{4}{x_2}\binom{4}{x_3}\binom{4}{x_4}\binom{36}{12 - x_1 - x_2 - x_3 - x_4} \Big/ \binom{52}{12}}{\binom{4}{x_1}\binom{4}{x_3}\binom{44}{12 - x_1 - x_3} \Big/ \binom{52}{12}}$$

$$= \frac{\binom{4}{x_2}\binom{4}{x_4}\binom{36}{12 - x_1 - x_2 - x_3 - x_4}}{\binom{44}{12 - x_1 - x_3}},$$

where $x_i = 0, 1, \ldots, 4$ and $x_2 + x_4 \le 12 - x_1 - x_3$. ////

3.2 Conditional Distribution Functions for Continuous Random Variables

Definition 13 Conditional probability density function Let X and Y be jointly continuous random variables with joint probability density function $f_{X,Y}(x, y)$. The *conditional probability density function* of Y given $X = x$, denoted by $f_{Y|X}(\cdot|x)$, is defined to be

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x, y)}{f_X(x)} \qquad (7)$$

if $f_X(x) > 0$, where $f_X(x)$ is the marginal probability density of X, and is undefined at points when $f_X(x) = 0$.
Similarly,