using the fact that the marginal density $f_{X_i}(x_i)$ is obtained from the joint density by

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{X_1,\ldots,X_k}(x_1, \ldots, x_k)\, dx_1 \ldots dx_{i-1} \cdot dx_{i+1} \ldots dx_k. \qquad ////$$

Similarly, the following theorem can be proved.

Theorem 5 If $g(x_1, \ldots, x_k) = (x_i - \mathscr{E}[X_i])^2$, then

$$\mathscr{E}[g(X_1, \ldots, X_k)] = \mathscr{E}[(X_i - \mathscr{E}[X_i])^2] = \text{var}\,[X_i]. \qquad ////$$

We might note that the "expectation" in the notation $\mathscr{E}[X_i]$ of Eq. (20) has two different interpretations; one is that the expectation is taken over the joint distribution of $X_1, \ldots, X_k$, and the other is that the expectation is taken over the marginal distribution of X_i. What Theorem 4 really says is that these two expectations are equivalent, and hence we are justified in using the same notation for both.

EXAMPLE 18 Consider the experiment of tossing two tetrahedra. Let X be the number on the first and Y the larger of the two numbers. We gave the joint discrete density function of X and Y in Example 2.

$$\begin{aligned}\mathscr{E}[XY] &= \sum xy f_{X,Y}(x, y)\\ &= 1 \cdot 1(\tfrac{1}{16}) + 1 \cdot 2(\tfrac{1}{16}) + 1 \cdot 3(\tfrac{1}{16}) + 1 \cdot 4(\tfrac{1}{16})\\ &\quad + 2 \cdot 2(\tfrac{2}{16}) + 2 \cdot 3(\tfrac{1}{16}) + 2 \cdot 4(\tfrac{1}{16}) + 3 \cdot 3(\tfrac{3}{16})\\ &\quad + 3 \cdot 4(\tfrac{1}{16}) + 4 \cdot 4(\tfrac{4}{16}) = \tfrac{135}{16}.\\ \mathscr{E}[X + Y] &= (1 + 1)\tfrac{1}{16} + (1 + 2)\tfrac{1}{16} + (1 + 3)\tfrac{1}{16} + (1 + 4)\tfrac{1}{16}\\ &\quad + (2 + 2)\tfrac{2}{16} + (2 + 3)\tfrac{1}{16} + (2 + 4)\tfrac{1}{16} + (3 + 3)\tfrac{3}{16}\\ &\quad + (3 + 4)\tfrac{1}{16} + (4 + 4)\tfrac{4}{16} = \tfrac{90}{16}.\end{aligned}$$

$\mathscr{E}[X] = \frac{5}{2}$, and $\mathscr{E}[Y] = \frac{50}{16}$; hence $\mathscr{E}[X + Y] = \mathscr{E}[X] + \mathscr{E}[Y]$. ////

EXAMPLE 19 Suppose $f_{X,Y}(x, y) = (x + y)I_{(0,1)}(x)I_{(0,1)}(y)$.

$$\mathscr{E}[XY] = \int_0^1 \int_0^1 xy(x + y)\, dx\, dy = \tfrac{1}{3}.$$

$$\mathscr{E}[X + Y] = \int_0^1 \int_0^1 (x + y)(x + y)\, dx\, dy = \tfrac{7}{6}.$$

$$\mathscr{E}[X] = \mathscr{E}[Y] = \tfrac{7}{12}. \qquad ////$$

EXAMPLE 20 Let the three-dimensional random variable (X_1, X_2, X_3) have the density

$$f_{X_1,X_2,X_3}(x_1, x_2, x_3) = 8x_1x_2x_3 I_{(0,1)}(x_1)I_{(0,1)}(x_2)I_{(0,1)}(x_3).$$

Suppose we want to find (i) $\mathscr{E}[3X_1 + 2X_2 + 6X_3]$, (ii) $\mathscr{E}[X_1X_2X_3]$, and (iii) $\mathscr{E}[X_1X_2]$. For (i) we have $g(x_1, x_2, x_3) = 3x_1 + 2x_2 + 6x_3$ and obtain

$$\mathscr{E}[g(X_1, X_2, X_3)] = \mathscr{E}[3X_1 + 2X_2 + 6X_3]$$
$$= \int_0^1 \int_0^1 \int_0^1 (3x_1 + 2x_2 + 6x_3)8x_1x_2x_3\, dx_1\, dx_2\, dx_3 = \tfrac{22}{3}.$$

For (ii), we get

$$\mathscr{E}[X_1X_2X_3] = \int_0^1 \int_0^1 \int_0^1 8x_1^2x_2^2x_3^2\, dx_1\, dx_2\, dx_3 = \tfrac{8}{27},$$

and for (iii) we get $\mathscr{E}[X_1X_2] = \frac{4}{9}$. ////

The following remark, the proof of which is left to the reader, displays a property of joint expectation. It is a generalization of (ii) in Theorem 3 of Chap. II.

Remark $\mathscr{E}\left[\sum_1^m c_i g_i(X_1, \ldots, X_k)\right] = \sum_1^m c_i \mathscr{E}[g_i(X_1, \ldots, X_k)]$ for constants $c_1, c_2, \ldots, c_m$. ////

4.2 Covariance and Correlation Coefficient

Definition 19 Covariance Let X and Y be any two random variables defined on the same probability space. The *covariance* of X and Y, denoted by cov $[X, Y]$ or $\sigma_{X,Y}$, is defined as

$$\text{cov}\,[X, Y] = \mathscr{E}[(X - \mu_X)(Y - \mu_Y)] \tag{21}$$

provided that the indicated expectation exists. ////

Definition 20 Correlation coefficient The *correlation coefficient*, denoted by $\rho[X, Y]$ or $\rho_{X,Y}$, of random variables X and Y is defined to be

$$\rho_{X,Y} = \frac{\text{cov}\,[X, Y]}{\sigma_X \sigma_Y} \tag{22}$$

provided that cov $[X, Y]$, σ_X, and σ_Y exist, and $\sigma_X > 0$ and $\sigma_Y > 0$. ////

Remark $\operatorname{cov}[X, Y] = \mathscr{E}[(X-\mu_X)(Y-\mu_Y)] = \mathscr{E}[XY] - \mu_X \mu_Y$.

PROOF
$$\begin{aligned}\mathscr{E}[(X-\mu_X)(Y-\mu_Y)] &= \mathscr{E}[XY - \mu_X Y - \mu_Y X + \mu_X \mu_Y] \\ &= \mathscr{E}[XY] - \mu_X \mathscr{E}[Y] - \mu_Y \mathscr{E}[X] + \mu_X \mu_Y \\ &= \mathscr{E}[XY] - \mu_X \mu_Y. \qquad ////\end{aligned}$$

EXAMPLE 21 Find $\rho_{X,Y}$ for X, the number on the first, and Y, the larger of the two numbers, in the experiment of tossing two tetrahedra. We would expect that $\rho_{X,Y}$ is positive since when X is large, Y tends to be large too. We calculated $\mathscr{E}[XY]$, $\mathscr{E}[X]$, and $\mathscr{E}[Y]$ in Example 18 and obtained $\mathscr{E}[XY] = \frac{135}{16}$, $\mathscr{E}[X] = \frac{5}{2}$, and $\mathscr{E}[Y] = \frac{50}{16}$. Thus $\operatorname{cov}[X, Y] = \frac{135}{16} - \frac{5}{2} \cdot \frac{50}{16} = \frac{10}{16}$. Now $\mathscr{E}[X^2] = \frac{30}{4}$ and $\mathscr{E}[Y^2] = \frac{170}{16}$; hence $\operatorname{var}[X] = \frac{5}{4}$ and $\operatorname{var}[Y] = \frac{55}{64}$. So,

$$\rho_{X,Y} = \frac{\frac{10}{16}}{\sqrt{\frac{5}{4}}\sqrt{\frac{55}{64}}} = \frac{2}{\sqrt{11}}. \qquad ////$$

EXAMPLE 22 Find $\rho_{X,Y}$ for X and Y if $f_{X,Y}(x, y) = (x+y)I_{(0,1)}(x)I_{(0,1)}(y)$. We saw that $\mathscr{E}[XY] = \frac{1}{3}$ and $\mathscr{E}[X] = \mathscr{E}[Y] = \frac{7}{12}$ in Example 19. Now $\mathscr{E}[X^2] = \mathscr{E}[Y^2] = \frac{5}{12}$; hence $\operatorname{var}[X] = \operatorname{var}[Y] = \frac{11}{144}$. Finally

$$\rho_{X,Y} = \frac{\frac{1}{3} - \frac{49}{144}}{\frac{11}{144}} = -\frac{1}{11}.$$

4.3 Conditional Expectations

In the following chapters we shall have occasion to find the expected value of random variables in conditional distributions, or the expected value of one random variable given the value of another.

Definition 21 **Conditional expectation** Let (X, Y) be a two-dimensional random variable and $g(\cdot,\cdot)$, a function of two variables. The *conditional expectation* of $g(X, Y)$ given $X = x$, denoted by $\mathscr{E}[g(X, Y) \mid X = x]$, is defined to be

$$\mathscr{E}[g(X, Y) \mid X = x] = \int_{-\infty}^{\infty} g(x, y) f_{Y|X}(y \mid x)\, dy \qquad (23)$$

if (X, Y) are jointly continuous, and