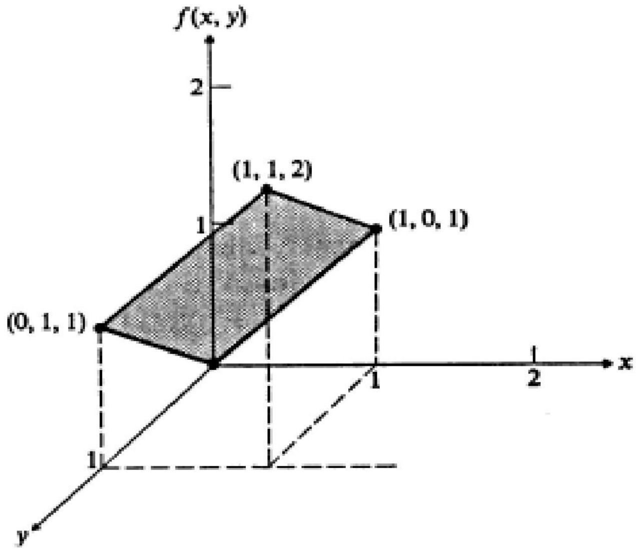

FIGURE 6

for $K = 1$. So $f(x, y) = (x + y)I_{(0,1)}(x)I_{(0,1)}(y)$ is a joint probability density function. It is sketched in Fig. 6.

Probabilities of events defined in terms of the random variables can be obtained by integrating the joint probability density function over the indicated region; for example

$$\begin{aligned} P[0 < X < \tfrac{1}{2}; 0 < Y < \tfrac{1}{4}] &= \int_0^{\frac{1}{4}} \int_0^{\frac{1}{2}} (x + y)\, dx\, dy \\ &= \int_0^{\frac{1}{4}} \left(\frac{1}{8} + \frac{y}{2}\right) dy \\ &= \frac{1}{32} + \frac{1}{64} \\ &= \frac{3}{64}, \end{aligned}$$

which is the volume under the surface $z = x + y$ over the region $\{(x, y): 0 < x < \frac{1}{2}; 0 < y < \frac{1}{4}\}$ in the xy plane. ////

Theorem 2 If X and Y are jointly continuous random variables, then knowledge of $F_{X,Y}(\cdot, \cdot)$ is equivalent to knowledge of an $f_{X,Y}(\cdot, \cdot)$. The remark extends to k-dimensional continuous random variables.

PROOF For a given $f_{X,Y}(\cdot, \cdot)$, $F_{X,Y}(x, y)$ is obtained for any (x, y) by

$$F_{X,Y}(x, y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f_{X,Y}(u, v)\, du\, dv.$$

For given $F_{X,Y}(\cdot,\cdot)$, an $f_{X,Y}(x, y)$ can be obtained by

$$f_{X,Y}(x, y) = \frac{\partial^2 F_{X,Y}(x, y)}{\partial x\, \partial y}$$

for x, y points, where $F_{X,Y}(x, y)$ is differentiable. ////

Definition 9 Marginal probability density functions If X and Y are jointly continuous random variables, then $f_X(\cdot)$ and $f_Y(\cdot)$ are called *marginal probability density functions*. More generally, let $X_{i_1}, \ldots, X_{i_m}$ be any subset of the jointly continuous random variables $X_1, \ldots, X_k$. $f_{X_{i_1}, \ldots, X_{i_m}}(x_{i_1}, \ldots, x_{i_m})$ is called a *marginal density of the* m-dimensional random variable $(X_{i_1}, \ldots, X_{i_m})$. ////

Remark If $X_1, \ldots, X_k$ are jointly continuous random variables, then any marginal probability density function can be found. (However, knowledge of all marginal densities does not, in general, imply knowledge of the joint density, as Example 8 below shows.) If X and Y are jointly continuous, then

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dy \qquad \text{and} \qquad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dx \tag{3}$$

since

$$f_X(x) = \frac{dF_X(x)}{dx} = \frac{d}{dx}\left[\int_{-\infty}^{x}\left(\int_{-\infty}^{\infty} f_{X,Y}(u, y)\, dy\right) du\right] = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dy.$$

////

EXAMPLE 7 Consider the joint probability density

$$f_{X,Y}(x, y) = (x + y)I_{(0,1)}(x)I_{(0,1)}(y).$$

$$\begin{aligned}
F_{X,Y}(x, y) = {} & I_{(0,1)}(x)I_{(0,1)}(y)\int_0^y\int_0^x (u + v)\, du\, dv \\
& + I_{(0,1)}(x)I_{[1,\infty)}(y)\int_0^1\int_0^x (u + v)\, du\, dv \\
& + I_{[1,\infty)}(x)I_{(0,1)}(y)\int_0^y\int_0^1 (u + v)\, du\, dv \\
& + I_{[1,\infty)}(x)I_{[1,\infty)}(y) \\
= {} & \tfrac{1}{2}\{(x^2y + xy^2)I_{(0,1)}(x)I_{(0,1)}(y) + (x^2 + x)I_{(0,1)}(x)I_{[1,\infty)}(y) \\
& + (y + y^2)I_{[1,\infty)}(x)I_{(0,1)}(y)\} + I_{[1,\infty)}(x)I_{[1,\infty)}(y).
\end{aligned}$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dy$$

$$= I_{(0,1)}(x) \int_0^1 (x + y)\, dy$$

$$= (x + \tfrac{1}{2}) I_{(0,1)}(x);$$

or,

$$f_X(x) = \frac{\partial F_{X,Y}(x, \infty)}{\partial x}$$

$$= \frac{\partial F_X(x)}{\partial x}$$

$$= I_{(0,1)}(x) \frac{\partial}{\partial x}\left(\frac{x^2 + x}{2}\right)$$

$$= (x + \tfrac{1}{2}) I_{(0,1)}(x).$$ ////

EXAMPLE 8 Let $f_X(x)$ and $f_Y(y)$ be two probability density functions with corresponding cumulative distribution functions $F_X(x)$ and $F_Y(y)$, respectively. For $-1 \le \alpha \le 1$, define

$$f_{X,Y}(x, y; \alpha) = f_X(x) f_Y(y)\{1 + \alpha[2F_X(x) - 1][2F_Y(y) - 1]\}. \qquad (4)$$

We will show (i) that for each α satisfying $-1 \le \alpha \le 1$, $f_{X,Y}(x, y; \alpha)$ is a joint probability density function and (ii) that the marginals of $f_{X,Y}(x, y; \alpha)$ are $f_X(x)$ and $f_Y(y)$, respectively. Thus, $\{f_{X,Y}(x, y; \alpha): -1 \le \alpha \le 1\}$ will be an infinite family of joint probability density functions, each having the

$$f_X(x) f_Y(y)\{1 + \alpha[2F_X(x) - 1][2F_Y(y) - 1]\} \ge 0$$
$$\text{if } 1 \ge -\alpha[2F_X(x) - 1][2F_Y(y) - 1];$$

but α, $2F_X(x) - 1$, and $2F_Y(y) - 1$ are all between -1 and 1, and hence also their product, which implies $f_{X,Y}(x, y; \alpha)$ is nonnegative. Since

$$1 = \int_{-\infty}^{\infty} f_X(x)\, dx = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f_{X,Y}(x, y; \alpha)\, dy\right) dx,$$