Generated by Foxit PDF Creator © Foxit Software
http://www.foxitsoftware.com For evaluation only.

JOINT AND CONDITIONAL DISTRIBUTIONS, STOCHASTIC INDEPENDENCE, MORE EXPECTATION

JOINT DISTRIBUTION FUNCTIONS

2.1 Cumulative Distribution Function

Definition 1 Joint cumulative distribution function Let $X_1, X_2, \ldots, X_k$ be k random variables all defined on the same probability space $(\Omega, \mathscr{A}, P[\cdot])$. The *joint cumulative distribution function* of $X_1, \ldots, X_k$, denoted by $F_{X_1, \ldots, X_k}(\cdot, \ldots, \cdot)$, is defined as $P[X_1 \le x_1; \ldots; X_k \le x_k]$ for all $(x_1, x_2, \ldots, x_k)$. ////

Thus a joint cumulative distribution function is a function with domain euclidean k space and counterdomain the interval [0, 1]. If $k = 2$, the joint cumulative distribution function is a function of two variables, and so its

of the downturned numbers. The goal is to find $F_{X,Y}(\cdot, \cdot)$, the joint cumulative distribution function of X and Y. Observe first that the random variables X and Y jointly take on only the values

(1, 1), (1, 2), (1, 3), (1, 4),
(2, 2), (2, 3), (2, 4),
(3, 3), (3, 4),
(4, 4).

(The first component is the value of X, and the second the value of Y.)

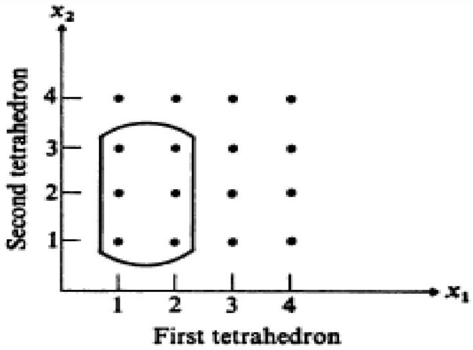

FIGURE 1
Sample space for experiment of tossing two tetrahedra.

Generated by Foxit PDF Creator © Foxit Software
http://www.foxitsoftware.com For evaluation only.

The sample space for this experiment is displayed in Fig. 1. The 16 sample points are assumed to be equally likely. Our objective is to find $F_{X,Y}(x, y)$ for each point (x, y). As an example let $(x, y) = (2, 3)$, and find $F_{X,Y}(2, 3) = P[X \le 2; \; Y \le 3]$. Now the event $\{X \le 2 \text{ and } Y \le 3\}$ corresponds to the encircled sample points in Fig. 1; hence $F_{X,Y}(2, 3) = \frac{6}{16}$. Similarly, $F_{X,Y}(x, y)$ can be found for other values of x and y. $F_{X,Y}(x, y)$ is tabled in Fig. 2. ////

We saw that the cumulative distribution function of a unidimensional random variable had certain properties; the same is true of a joint cumulative. We shall list these properties for the joint cumulative distribution function of two random variables; the generalization to k dimensions is straightforward.

TABLE OF VALUES OF $F_{X,Y}(x,y)$

$4 \le y$	0	$\frac{4}{16}$	$\frac{8}{16}$	$\frac{12}{16}$	1
$3 \le y < 4$	0	$\frac{3}{16}$	$\frac{6}{16}$	$\frac{9}{16}$	$\frac{9}{16}$
$2 \le y < 3$	0	$\frac{2}{16}$	$\frac{4}{16}$	$\frac{4}{16}$	$\frac{4}{16}$
$1 \le y < 2$	0	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$
$y < 1$	0	0	0	0	0
	$x < 1$	$1 \le x < 2$	$2 \le x < 3$	$3 \le x < 4$	$4 \le x$

Properties of bivariate cumulative distribution function $F(\cdot, \cdot)$

(i) $F(-\infty, y) = \lim_{x \to -\infty} F(x, y) = 0$ for all y, $F(x, -\infty) = \lim_{y \to -\infty} F(x, y) = 0$ for all x, and $\lim_{\substack{x \to \infty \\ y \to \infty}} F(x, y) = F(\infty, \infty) = 1$.

(ii) If $x_1 < x_2$ and $y_1 < y_2$, then $P[x_1 < X \le x_2; y_1 < Y \le y_2]$ $= F(x_2, y_2) - F(x_2, y_1) - F(x_1, y_2) + F(x_1, y_1) \ge 0$.

(iii) $F(x, y)$ is right continuous in each argument; that is, $\lim_{0 < h \to 0} F(x + h, y) = \lim_{0 < h \to 0} F(x, y + h) = F(x, y)$.

We will not prove these properties. Property (ii) is a *monotonicity* property of sorts; it is not equivalent to $F(x_1, y_1) \le F(x_2, y_2)$ for $x_1 \le x_2$ and $y_1 \le y_2$.

Generated by Foxit PDF Creator © Foxit Software
http://www.foxitsoftware.com For evaluation only.

Definition 2 Bivariate cumulative distribution function Any function satisfying properties (i) to (iii) is defined to be a *bivariate cumulative distribution function* without reference to any random variables. ////

Definition 3 Marginal cumulative distribution function If $F_{X,Y}(\cdot, \cdot)$ is the joint cumulative distribution function of X and Y, then the cumulative distribution functions $F_X(\cdot)$ and $F_Y(\cdot)$ are called *marginal cumulative distribution functions.* ////

Remark $F_X(x) = F_{X,Y}(x, \infty)$, and $F_Y(y) = F_{X,Y}(\infty, y)$; that is, knowledge of the joint cumulative distribution function of X and Y implies knowledge of the two marginal cumulative distribution functions. ////

The converse of the above remark is not generally true; in fact, an example (Example 8) will be given in Subsec. 2.3 below that gives an entire family of joint cumulative distribution functions, and each member of the family has the same marginal distributions.

We will conclude this section with a remark that gives an inequality involving the joint cumulative distribution and marginal distributions. The proof is left as an exercise.

Remark $F_X(x) + F_Y(y) - 1 \leq F_{X,Y}(x, y) \leq \sqrt{F_X(x)F_Y(y)}$ for all x, y. ////

2.2 Joint Density Functions for Discrete Random Variables

If $X_1, X_2, \ldots, X_k$ are random variables defined on the same probability space, then $(X_1, X_2, \ldots, X_k)$ is called a *k-dimensional random variable*.