

Lecture #6

- **Strain energy.**
- **Yielding criteria for ductile metals.**
- **Maximum shear stress or Tresca criterion.**
- **The distortion energy theory (von.Mises theory).**
- **Maximum Octahedral shear stress theory.**
- **Examples.**

References:

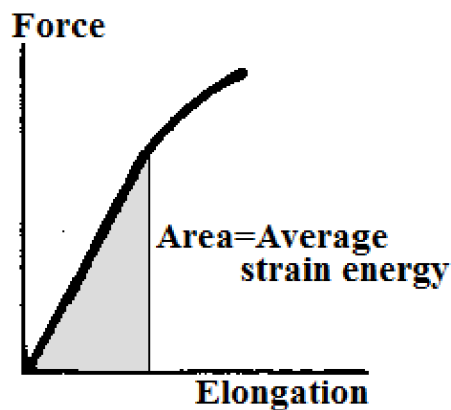
- 1-Norton, 1998, An Integrated approach to machine design.
- 2- Hosford W.F, (2005), Mechanical Behavior of Materials, Cambridge
- 3- Meyers M.A. and Chawla K.K., (2009). Mechanical Behavior of Materials, Prentice- Hall.
- 4- Dieter G.E., (1986), Mechanical Metallurgy, McGraw-Hill
- 5- Michael F. Ashby, David R. H. Jones, 1999, Engineering Materials 2, An Introduction to Microstructures, Processing and Design, Second Edition, Butterworth&Heinemann.

Lecture # 6

Strain energy:

The elastic strain energy U is the energy expended by the action of external forces in deforming an elastic body. All the work performed during elastic deformation is stored as elastic energy, and this energy is recovered on the release of the applied forces. In the deformation of an elastic body, the average strain energy is:

$$U = \frac{1}{2} P \delta$$



For an elemental cube subjected to only a tensile stress along x-axis, the elastic strain energy is:

$$dU = \frac{1}{2} (\sigma_x A) (\epsilon_x dx) = \frac{1}{2} (\sigma_x \epsilon_x) (A dx)$$

But $(A dx)$ is the volume of the element, so the strain energy per unit volume or strain energy density U_0 is:

$$U_0 = \frac{1}{2} \sigma_x \epsilon_x = \frac{1}{2} \frac{\sigma_x^2}{E} = \frac{1}{2} \epsilon_x^2 E$$

Note that the lateral strains which accompany deformation in simple tension do not enter into the expression for strain energy because forces do not exist in the direction of the lateral strains.

By the same type of reasoning, the strain energy per unit volume of an element subjected to *pure shear* is given by

$$U_0 = \frac{1}{2} \tau_{xy} \gamma_{xy} = \frac{1}{2} \frac{\tau_{xy}^2}{G} = \frac{1}{2} \gamma_{xy}^2 G$$

The elastic strain energy for a general three-dimensional stress

$$U_0 = \frac{1}{2}(\sigma_x \epsilon_x + \sigma_y \epsilon_y + \sigma_z \epsilon_z + \tau_{xy} \gamma_{xy} + \tau_{xz} \gamma_{xz} + \tau_{yz} \gamma_{yz})$$

But:

$$\epsilon_x = \frac{1}{E} [\sigma_x - \nu(\sigma_y + \sigma_z)]; \epsilon_y = \frac{1}{E} [\sigma_y - \nu(\sigma_z + \sigma_x)]$$

$$\text{and } \epsilon_z = \frac{1}{E} [\sigma_z - \nu(\sigma_x + \sigma_y)] \quad \text{also:}$$

$$\tau_{xy} = G\gamma_{xy} \quad \tau_{yz} = G\gamma_{yz} \quad \tau_{xz} = G\gamma_{xz}$$

So total strain energy per unit volume:

$$U_0 = \frac{1}{2E} (\sigma_x^2 + \sigma_y^2 + \sigma_z^2) - \frac{\nu}{E} (\sigma_x \sigma_y + \sigma_y \sigma_z + \sigma_x \sigma_z) + \frac{1}{2G} (\tau_{xy}^2 + \tau_{xz}^2 + \tau_{yz}^2)$$

Recalling the relation between the stress and strain (in matrix form, **lect. 5 page 6**) and putting:

$$\frac{\nu E}{(1 + \nu)(1 - 2\nu)} = \lambda \quad (\text{called Lamé's constant})$$

and:

$$\Delta = \epsilon_x + \epsilon_y + \epsilon_z$$

the strain energy can be expressed in terms of strains and the elastic constants as:

$$U_0 = \frac{1}{2}\lambda\Delta^2 + G(\epsilon_x^2 + \epsilon_y^2 + \epsilon_z^2) + \frac{1}{2}G(\gamma_{xy}^2 + \gamma_{xz}^2 + \gamma_{yz}^2)$$

The derivative of U_0 with respect to any strain (or stress) component gives the corresponding stress (or strain) component, so:

$$\frac{\partial U_0}{\partial \epsilon_x} = \lambda\Delta + 2G\epsilon_x = \sigma_x \quad \text{and} \quad \partial U_0 / \partial \sigma_x = \epsilon_x$$

Yielding criteria for ductile metals (Theories of failure by yielding):

According to each theory of failure, the selected criteria of failure for the complex stress state are equated to the value of the same criteria corresponding to failure in a simple tensile test (simple uniaxial system). The hydrostatic stress does not cause yielding and as a result this stress does not influence the stress at which yielding occurs. Therefore, we look for the stress deviator to be involved with yielding. Moreover, for an isotropic material, the yielding criterion must be independent of the choice of axis, i.e., it must be an invariant function. So the yield criteria must be some function of the invariants of the stress deviator. There are two accepted criteria for ductile metals:

Maximum shear stress or Tresca criterion:

This yield criteria states that yielding occurs when the maximum shear stress reaches the value of the shear stress in the uniaxial- tension test. So safety is obtained whenever:

$$\tau_{allowable} = \frac{\tau_{yieldpoint}}{n} = \frac{\sigma_{yieldpoint}}{2n}$$

For triaxial stress state when $\sigma_1 > \sigma_2 > \sigma_3$:

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{\sigma_{yieldpoint}}{2}$$

So the maximum shear stress criterion is given by: $\sigma_{yieldpoint} = (\sigma_1 - \sigma_3)$

For biaxial stress state when $\sigma_1 > \sigma_2$ and $\sigma_3 = 0$:

- For like stresses, i.e. both σ_1 & σ_2 are tension or compression (1st and 3rd quadrants):

$$\tau_{max} = \frac{\sigma_1 - 0}{2} \text{ or } \frac{\sigma_2 - 0}{2} = \frac{\sigma_{yieldpoint}}{2}$$

i.e. $\sigma_1 = \sigma_{yieldpoint}$ or $\sigma_2 = \sigma_{yieldpoint}$

- For unlike stresses (2nd and 4th quadrants) :

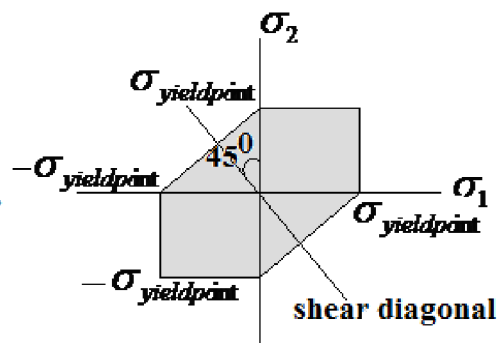
$$\tau_{\max} = \frac{\sigma_1 - \sigma_2}{2} = \frac{\sigma_{\text{yieldpoint}}}{2}$$

or

$$\tau_{\max} = \frac{\sigma_2 - \sigma_1}{2} = \frac{\sigma_{\text{yieldpoint}}}{2}$$

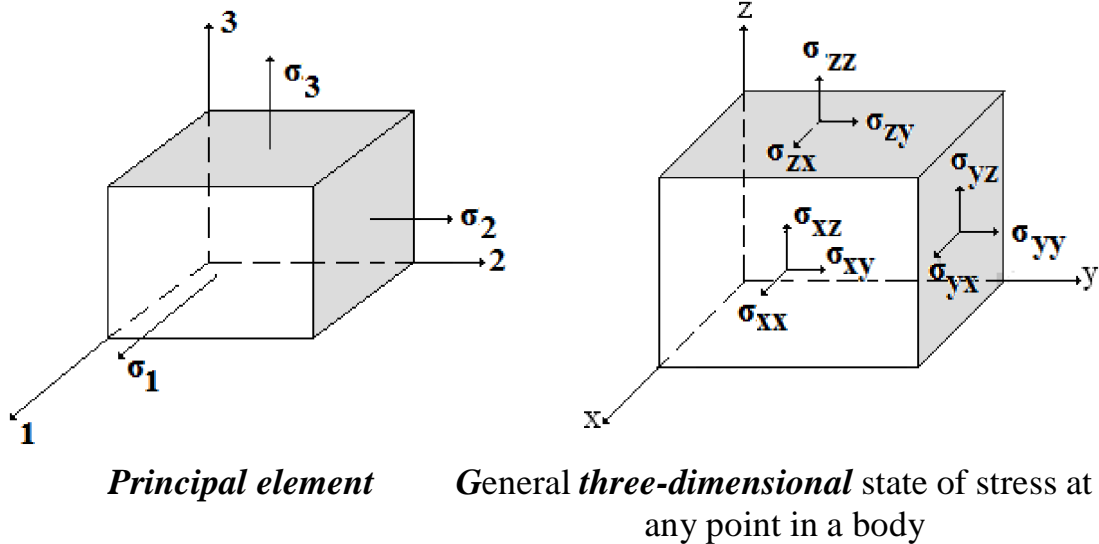
So $\frac{\sigma_1}{\sigma_{\text{yieldpoint}}} - \frac{\sigma_2}{\sigma_{\text{yieldpoint}}} = 1$ or $\frac{\sigma_2}{\sigma_{\text{yieldpoint}}} - \frac{\sigma_1}{\sigma_{\text{yieldpoint}}} = 1$

These are straight lines and produce the failure envelope of the fig. shown.



The distortion energy theory (von.Mises theory):

Also called shear energy theory.

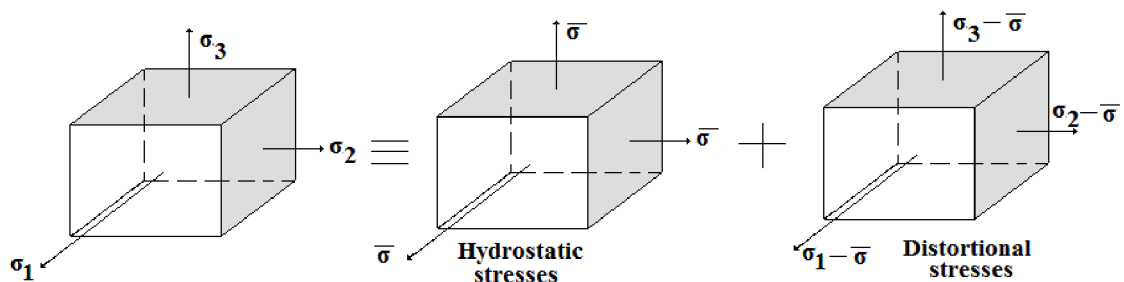


Total strain energy per unit volume for general *three-dimensional* state of stress:

$$U_0 = \frac{1}{2E} (\sigma_x^2 + \sigma_y^2 + \sigma_z^2) - \frac{\nu}{E} (\sigma_x \sigma_y + \sigma_y \sigma_z + \sigma_x \sigma_z) + \frac{1}{2G} (\tau_{xy}^2 + \tau_{xz}^2 + \tau_{yz}^2)$$

Total strain energy per unit volume for *Principal element in three-dimensional* state of stress:

$$U_0 = \frac{1}{2E} (\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu (\sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_1 \sigma_3))$$



1- Volumetric strain energy (associated with the hydrostatic stresses):

$$\sigma_1 = \sigma_2 = \sigma_3 = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$$

By substituting the above equation in the equation of total strain energy per unit volume for *Principal element in three-dimensional* state of stress:

$$U_{vol} = \frac{(1-2\nu)}{6E} [(\sigma_1 + \sigma_2 + \sigma_3)^2]$$

2- Shear strain energy per unit volume (U_s)/distortion of the shape due to shear stress:

$$U_s = U_0 - U_{vol}$$

$$U_s = \frac{(1+\nu)}{6E} [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] \dots(1)$$

or:

$$U_s = \frac{1}{6G} [(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - (\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)]$$

Yielding will occur when the strain energy of the distortion per unit volume equals that for a specimen in uniaxial tension or compression (strained to the yield point), so:

3- For the simple tension (or compression) test: $\sigma_1 = \sigma_{\text{yield point}}$ and $\sigma_2 = \sigma_3 = 0$, sub. in (1) get that the distortion energy is:

$$U_d = \frac{(1 + \nu)}{3E} \sigma_{yield} \dots\dots\dots (2)$$

Equating (1)&(2) get that:

$$2\sigma_{yield}^2 = (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2I_1^2 + 6I_2$$

$$\sigma_{yield} = \frac{1}{\sqrt{2}} \left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right]^{1/2}$$

$$\sigma_{yield} = \frac{1}{\sqrt{2}} \left[(\sigma_{xx} - \sigma_{yy})^2 + (\sigma_{yy} - \sigma_{zz})^2 + (\sigma_{zz} - \sigma_{xx})^2 + 6(\sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{zx}^2) \right]^{1/2}$$

For a biaxial stress state ($\sigma_3=0$):

$$\sigma_{yield}^2 = \sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2$$

For pure torsion $\sigma_2 = -\sigma_1$ and $\tau = \sigma_1$, sub. get that

$$\sigma_{yield} = \sqrt{3}\sigma_1 = \sqrt{3}\tau$$

or
$$\tau_{yield} = \frac{1}{\sqrt{3}}\sigma_{yield} = 0.577\sigma_{yield}$$

For analyses and design purposes, it is convenient to define a von Mises stress:

$$\bar{\sigma}_{VM} = \sqrt{\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2}$$

and in terms of the applied stresses:

$$\bar{\sigma}_{VM} = \sqrt{\sigma_x^2 + \sigma_y^2 - \sigma_x\sigma_y + 3\sigma_{xy}^2}$$

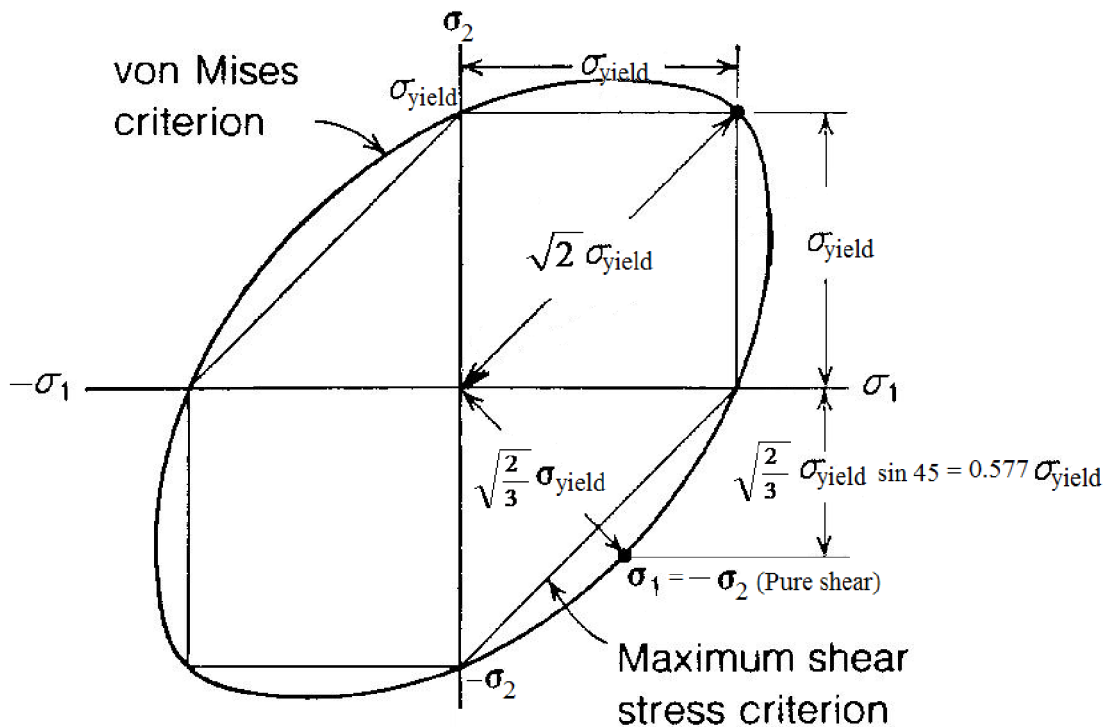
and
$$\bar{\sigma}_{VM} = \sigma_{yield} / n$$

For graphical representation:

$$\sigma_{\text{yield}}^2 = \sigma_1^2 - \sigma_1 \sigma_2 + \sigma_2^2$$

$$\left(\frac{\sigma_1}{\sigma_y}\right)^2 + \left(\frac{\sigma_2}{\sigma_y}\right)^2 - \left(\frac{\sigma_1}{\sigma_y}\right)\left(\frac{\sigma_2}{\sigma_y}\right) = 1$$

This equation represents ellipse with semi-axes $\sqrt{2} \sigma_{yp}$ and $\sqrt{\frac{2}{3}} \sigma_y$ at 45° to the coordinate axes as shown in the fig.



Maximum Octahedral shear stress theory:

$$\Rightarrow \tau_{oct} = \frac{1}{3} \left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right]^{1/2}$$

For uniaxial tension: $\sigma_1 = \sigma_{\text{yield point}}$ and $\sigma_2 = \sigma_3 = 0$, sub. and get that:

$$\tau_{oct} = \frac{\sqrt{2}}{3} \sigma_y$$

So the criterion of failure becomes:

$$\frac{\sqrt{2}}{3} \sigma_y = \frac{1}{3} \left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right]^{1/2}$$

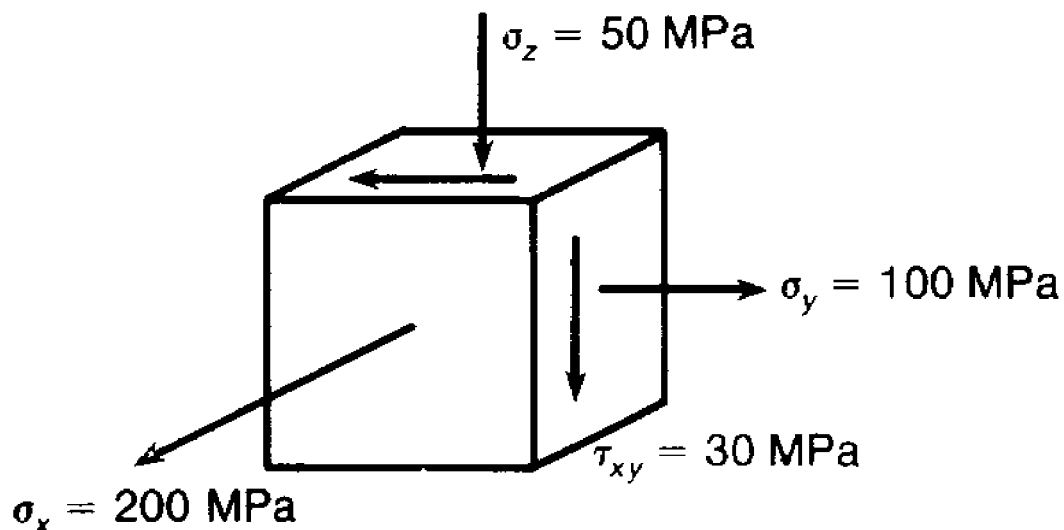
or

$$2\sigma_y^2 = \left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right] = 2I_1^2 + 6I_2$$

This is the same as that for von Mises distortion energy theory.

Example #1

For the state of stress shown, if the part is made from 7075-T6 Al-alloy ($\sigma_{yp} = 500\text{MPa}$), will it exhibit yielding? If not what is the safety factor? Base your answer on the distortion energy theory (von.Mises theory).



Solution:

According to von.Mises:

$$\sigma_{\text{yield}} = \frac{1}{\sqrt{2}} \left[(\sigma_{xx} - \sigma_{yy})^2 + (\sigma_{yy} - \sigma_{zz})^2 + (\sigma_{zz} - \sigma_{xx})^2 + 6(\sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{xz}^2) \right]^{1/2}$$

$$\sigma_{\text{YP}} = \frac{1}{\sqrt{2}} \left[(200 - 100)^2 + (100 - (-50))^2 + (-50 - 200)^2 + 6(30)^2 \right]^{1/2}$$

$$= 224 \text{MPa} < 500 \text{MPa}$$

So yielding will not occur and the safety factor, $n = 500/224 = 2.2$

Example #2

A material subjected to a simple tension test shows an elastic limit of 240MPa. Calculate the factor of safety provided if the principal stresses set up in a complex two dimensional stress system are limited to 140 MPa tensile and 45 MPa compressive. The appropriate theories of failure on which your answer should be based are:

- (a) the maximum shear stress theory;
- (b) The distortion energy theory (von.Mises theory).

Solution:

(a) Maximum shear stress theory: $\sigma_3 = 0$

$$\tau_{\text{max}} = \frac{\sigma_1 - \sigma_2}{2} = \frac{\sigma_{\text{yieldpoint}}}{2n}$$

$$\text{or } \sigma_1 - \sigma_2 = \frac{\sigma_{\text{yieldpoint}}}{n} \quad n = \frac{240}{140 - (-45)} = 1.297$$

(b) The distortion energy theory (von.Mises theory).

$$\frac{\sigma_{\text{yield}}}{n} = \sqrt{\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2}$$

$$\frac{240}{n} = \sqrt{140^2 - (140)(-45) + (-45)^2} \quad n = 1.436$$

Example #3

A steel tube has a mean diameter of 100mm and a thickness of 3 mm. Calculate the torque which can be transmitted by the tube with a factor of safety of **2.25** if the criterion of failure is

(a) maximum shear stress; (b) The distortion energy theory (von.Mises theory). Elastic limit of the steel in tension is **225** MPa.

Solution:

$$\tau = \frac{16Td_o}{\pi(d_o^4 - d_i^4)} = \frac{16T * 0.103}{\pi(0.103^4 - 0.097^4)} = 21.8T \text{ KN.m}$$

$$\sigma_1 = \tau ; \sigma_2 = -\tau ; \sigma_3 = 0$$

(a) maximum shear stress:

$$\sigma_1 - \sigma_2 = \frac{\sigma_{yieldpoint}}{n} \quad 21.8T + 21.8T = \frac{225}{2.25} \quad ; \quad T = 2.293 \text{ KPa}$$

(b) The distortion energy theory (von.Mises theory):

$$\frac{\sigma_{yield}}{n} = \sqrt{\sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y + 3\sigma_{xy}^2}$$

$$\frac{225 * 10^3}{2.25} = \sqrt{3(21.8T)^2} \quad T = 2.648 \text{ KN.m}$$