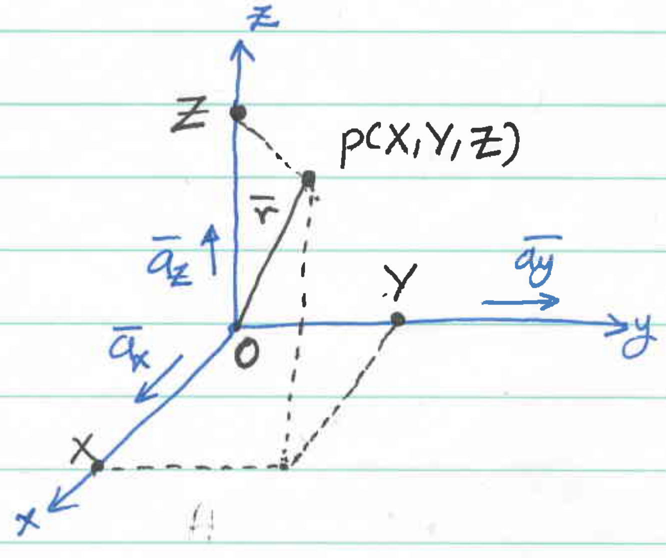
« The coordinate systems »

From a mathematical point of view it is very convenient to work with vectors when they are resolved into components along three mutually orthogonal (perpendicular) directions. We will mainly use three orthogonal coordinate systems :

- the rectangular (or Cartesian) coordinate system.
- the cylindrical (circular) coordinate system.
- the spherical coordinate system.

Rectangular Coordinate System:

A rectangular (Cartesian) coordinate system is asystem formed by three mutually orthogonal straight lines. The three straight lines are called the x, y and z axes. the point of intersection of these axes is the origion.



the unit vectors $\bar{a}_x$, $\bar{a}_y$ and $\bar{a}_z$ are indicate (2)
to the direction of the component of a vector along the x, y, and z axes. respectively.

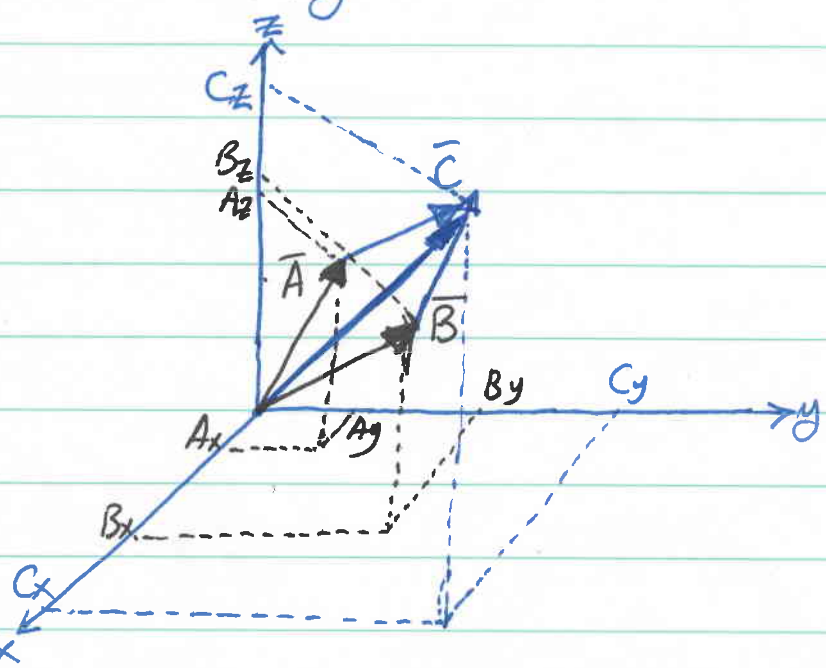
A point $P(X, Y, Z)$ in space can be uniquely defined by its projections on the three axes as illustrated in fig. above.

the Position Vector $\bar{r}$ can be expressed in terms of its component as :-

$$\bar{r} = X\,\bar{a}_x + Y\,\bar{a}_y + Z\,\bar{a}_z$$

where X, Y, and Z are the scalar Projections of $\bar{r}$ on the x, y, and z axes.

If A_x, A_y and A_z are the scalar Projections of $\bar{A}$ as shown in fig.



then $\bar{A}$ can be written as

$$\boxed{\bar{A} = A_x\,\bar{a}_x + A_y\,\bar{a}_y + A_z\,\bar{a}_z}$$

Similarly, We can express vector $\bar{B}$ as

$$\bar{B} = B_x \bar{a_x} + B_y \bar{a_y} + B_z \bar{a_z}$$

The sum of $\bar{A}$ and $\bar{B}$

$$\bar{C} = \bar{A} + \bar{B}$$

$$\bar{C} = (A_x + B_x)\bar{a_x} + (A_y + B_y)\bar{a_y} + (A_z + B_z)\bar{a_z}$$

$$= C_x \bar{a_x} + C_y \bar{a_y} + C_z \bar{a_z}$$

C_x, C_y and C_z are the Component of $\bar{C}$ along the $\bar{a_x}$, $\bar{a_y}$ and $\bar{a_z}$ unit vectors.

Since the three unit vectors are mutually orthogonal,

the dot product yields:

$$\bar{a_x} \cdot \bar{a_x} = 1 \qquad \bar{a_y} \cdot \bar{a_y} = 1 \qquad \bar{a_z} \cdot \bar{a_z} = 1$$

and

$$\bar{a_x} \cdot \bar{a_y} = \bar{a_y} \cdot \bar{a_z} = \bar{a_z} \cdot \bar{a_x} = 0$$

the cross product of the unit vectors yields:

$$\bar{a_x} \times \bar{a_x} = \bar{a_y} \times \bar{a_y} = \bar{a_z} \times \bar{a_z} = 0$$

and

$$\bar{a_x} \times \bar{a_y} = \bar{a_z} \qquad \bar{a_y} \times \bar{a_z} = \bar{a_x} \qquad \bar{a_z} \times \bar{a_x} = \bar{a_y}$$

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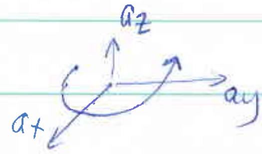


The dot product of vectors $\bar{A}$ and $\bar{B}$

$$\bar{A} \cdot \bar{B} = A_x B_x + A_y B_y + A_z B_z$$ as scalar

In Particular:

$$\bar{A} \cdot \bar{A} = A_x^2 + A_y^2 + A_z^2 \Rightarrow |A| = \sqrt{\bar{A} \cdot \bar{A}}$$

$$\therefore |A| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

Example: Given $\bar{A} = 3\bar{a}_x + 2\bar{a}_y - \bar{a}_z$ and
$\bar{B} = \bar{a}_x - 3\bar{a}_y + 2\bar{a}_z$

find $\bar{C}$ such that $\bar{C} = 2\bar{A} - 3\bar{B}$.
Find the unit vector $\bar{a}_c$ and the angle it makes with the Z axis.

Solution:

$$\bar{C} = 2\bar{A} - 3\bar{B}$$
$$= 2[3\bar{a}_x + 2\bar{a}_y - \bar{a}_z] - 3[\bar{a}_x - 3\bar{a}_y + 2\bar{a}_z]$$
$$= 3\bar{a}_x + 13\bar{a}_y - 8\bar{a}_z$$

$$\therefore |C| = \sqrt{(3)^2 + (13)^2 + (-8)^2} = 15.556$$

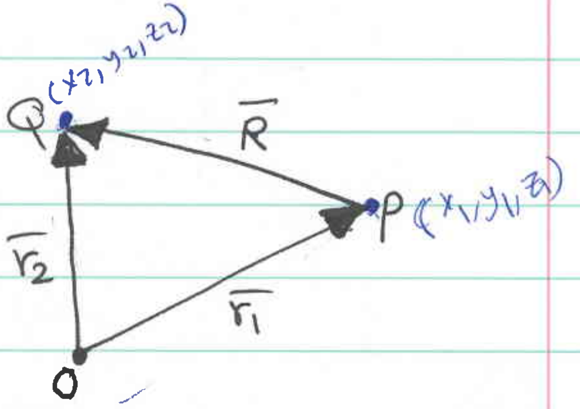
$$\therefore \bar{a}_c = \frac{\bar{C}}{|C|} = 0.193\bar{a}_x + 0.836\bar{a}_y - 0.514\bar{a}_z$$

The angle above unit vector makes with Z axis is

$$\theta_z = \cos^{-1}\left[\frac{C_z}{|C|}\right] = \cos^{-1}\left[\frac{-8}{15.556}\right] = 120.95^\circ$$

Example:- Find the vector $\overline{R}$ that is directed from a point $P(x_1, y_1, z_1)$ to point $Q(x_2, y_2, z_2)$.

Solution:



A vector from one point to another is referred to as a distance vector.

Let $\overline{r_1}$ and $\overline{r_2}$ be the position vectors of points P and Q as depicted in figure, then

$$\overline{r_1} = x_1 \overline{a_x} + y_1 \overline{a_y} + z_1 \overline{a_z}$$

and

$$\overline{r_2} = x_2 \overline{a_x} + y_2 \overline{a_y} + z_2 \overline{a_z}$$

The distance vector $\overline{R}$ from point P to Q is:-

$$\boxed{\overline{R} = \overline{r_2} - \overline{r_1}}$$

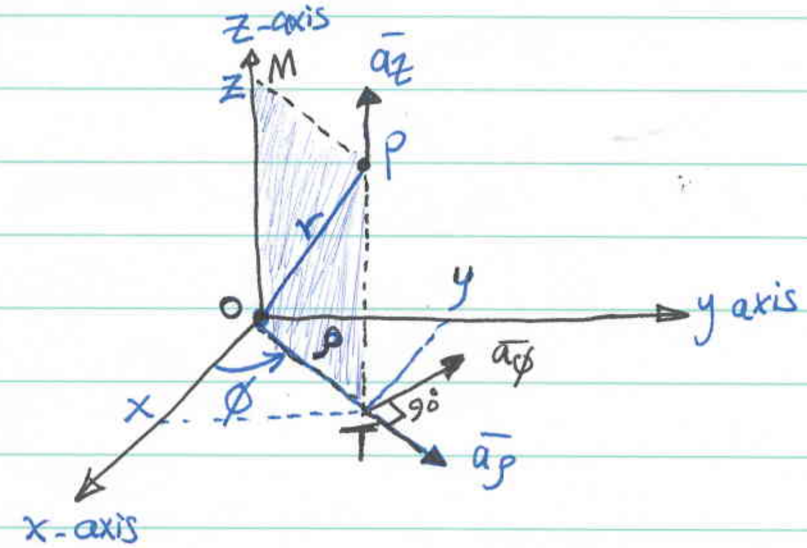
$$= (x_2 - x_1)\overline{a_x} + (y_2 - y_1)\overline{a_y} + (z_2 - z_1)\overline{a_z}$$

- The cross product of vectors $\overline{A}$ and $\overline{B}$ can be also computed as $\overline{C} = \overline{A} \times \overline{B}$

$$\overline{A} \times \overline{B} = \begin{vmatrix} \overline{a_x} & \overline{a_y} & \overline{a_z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Cylindrical Coordinate System.

A point $P(x, y, z)$ can also be represented in terms of ρ, ϕ, and z as shown in fig.



Where ρ is the projection of r on the xy-plane,
ϕ is the angle from the positive x-axis to the plane OTPM
z is the projection of r on the z-axis.

r = the distance from O to P

∴ ρ, ϕ and z represent the cylindrical coordinates ~~system~~ of point $P(\rho, \phi, z)$.

$$x = \rho \cos\phi$$
$$y = \rho \sin\phi$$

$$\rho = \sqrt{x^2 + y^2} = \text{constant}$$

It is coordinate surface and represent

a cylinder of radius ρ with the z-axis as its axis. from figure

$$0 \leq \rho \leq \infty$$

$\phi = \tan^{-1}\left(\frac{y}{x}\right) = \text{Constant.}$

is a plane hinged on the z-axis.

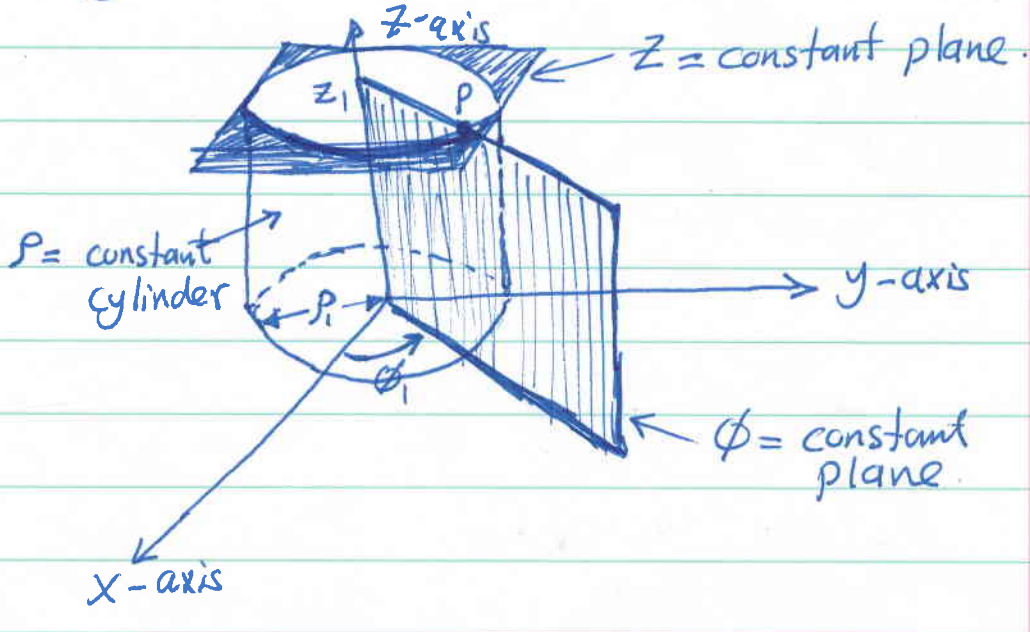
$z = \text{constant}$

is a plane parallel to the xy-plane.

The corresponding unit vector are: $\bar{a}_\rho$, $\bar{a}_\phi$ and $\bar{a}_z$.

the angle ϕ is measured with respect to the x-axis in the counterclockwise direction. Hence, ϕ varies from $(0$ to $2\pi)$

* Note that the $\bar{a}_\rho$ and $\bar{a}_\phi$ are not unidirectional; they change direction as ϕ increases or decreases.



Important Note:

If the two vectors $\bar{A}$ and $\bar{B}$ are defined at a common point $P(\rho, \phi, z)$ or in a $\phi =$ constant plane, we can add, subtract, and multiply these vectors as we did in the rectangular (cartesian) coordinate system.

For example: if the two vectors at Point $P(\rho, \phi, z)$ are

$$\bar{A} = A_\rho \bar{a}_\rho + A_\phi \bar{a}_\phi + A_z \bar{a}_z$$

and

$$\bar{B} = B_\rho \bar{a}_\rho + B_\phi \bar{a}_\phi + B_z \bar{a}_z$$

then

$$\bar{A} + \bar{B} = (A_\rho + B_\rho)\bar{a}_\rho + (A_\phi + B_\phi)\bar{a}_\phi + (A_z + B_z)\bar{a}_z$$

$$\bar{A} \cdot \bar{B} = A_\rho B_\rho + A_\phi B_\phi + A_z B_z$$

$$\bar{A} \times \bar{B} = \begin{vmatrix} \bar{a}_\rho & \bar{a}_\phi & \bar{a}_z \\ A_\rho & A_\phi & A_z \\ B_\rho & B_\phi & B_z \end{vmatrix}$$

The Dot and Cross Product of the Unit vectors in cylindrical coordinate system are:

$$\bar{a}_\rho \cdot \bar{a}_\rho = 1 \qquad \bar{a}_\rho \cdot \bar{a}_\phi = 0$$
$$\bar{a}_\phi \cdot \bar{a}_\phi = 1 \qquad \bar{a}_\phi \cdot \bar{a}_z = 0$$
$$\bar{a}_z \cdot \bar{a}_z = 1 \qquad \bar{a}_z \cdot \bar{a}_\rho = 0$$

$$\bar{a}_\rho \times \bar{a}_\rho = 0 \qquad \bar{a}_\rho \times \bar{a}_\phi = \bar{a}_z$$
$$\bar{a}_\phi \times \bar{a}_\phi = 0 \qquad \bar{a}_\phi \times \bar{a}_z = \bar{a}_\rho$$
$$\bar{a}_z \times \bar{a}_z = 0 \qquad \bar{a}_z \times \bar{a}_\rho = \bar{a}_\phi$$

Transformation of Unit Vectors

$$\begin{bmatrix} \bar{a}_\rho \\ \bar{a}_\phi \\ \bar{a}_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{a}_x \\ \bar{a}_y \\ \bar{a}_z \end{bmatrix}$$

$$\bar{a}_x \cdot \bar{a}_\rho = \cos\phi$$
$$\bar{a}_y \cdot \bar{a}_\rho = \sin\phi$$
$$\bar{a}_x \cdot \bar{a}_\phi = -\sin\phi$$
$$\bar{a}_y \cdot \bar{a}_\phi = \cos\phi$$

Transformation of a vector

If $\bar{A}$ is a vector given in the cylindrical coordinate system, it can be expressed in the Cartesian coordinate system by projecting it onto the x, y, and z axes.

That is, the scalar projection of $\bar{A}$ onto x-axis is

$$A_x = \bar{A} \cdot \bar{a}_x$$
$$= A_\rho \cdot \bar{a}_\rho \cdot \bar{a}_x + A_\phi \bar{a}_\phi \cdot \bar{a}_x + A_z \bar{a}_z \cdot \bar{a}_x$$
$$= A_\rho \cos\phi - A_\phi \sin\phi$$

Similarly

$$A_y = \bar{A} \cdot \bar{a}_y = A_\rho \sin\phi + A_\phi \cos\phi$$
$$A_z = \bar{A} \cdot \bar{a}_z = A_z$$

In matrix form :-

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

Similarly Procedure, but to a vector in rectangular and need cylindrical :-

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

Example 8 Write an expression for a position vector at any point in space in the rectangular coord. sys. Then transform the position vector into a vector in the cylin. coord. system.

Solution: The position vector of any point $P(x, y, z)$ in space is

$$\bar{A} = x\,\bar{a}_x + y\,\bar{a}_y + z\,\bar{a}_z$$

$$A_\rho = \bar{A} \cdot \bar{a}_\rho = x \cos\phi + y \sin\phi$$
$$A_\phi = \bar{A} \cdot \bar{a}_\phi = -x \sin\phi + y \cos\phi$$
$$A_z = \bar{A} \cdot \bar{a}_z \Rightarrow A_z = z$$

Substituting $x = \rho\cos\phi$
$y = \rho\sin\phi$

we obtain

$$A_\rho = \rho \;,\quad A_\phi = 0 \;,\quad A_z = z$$

Thus: $\bar{A}$ in the cylind. coord. sys. is

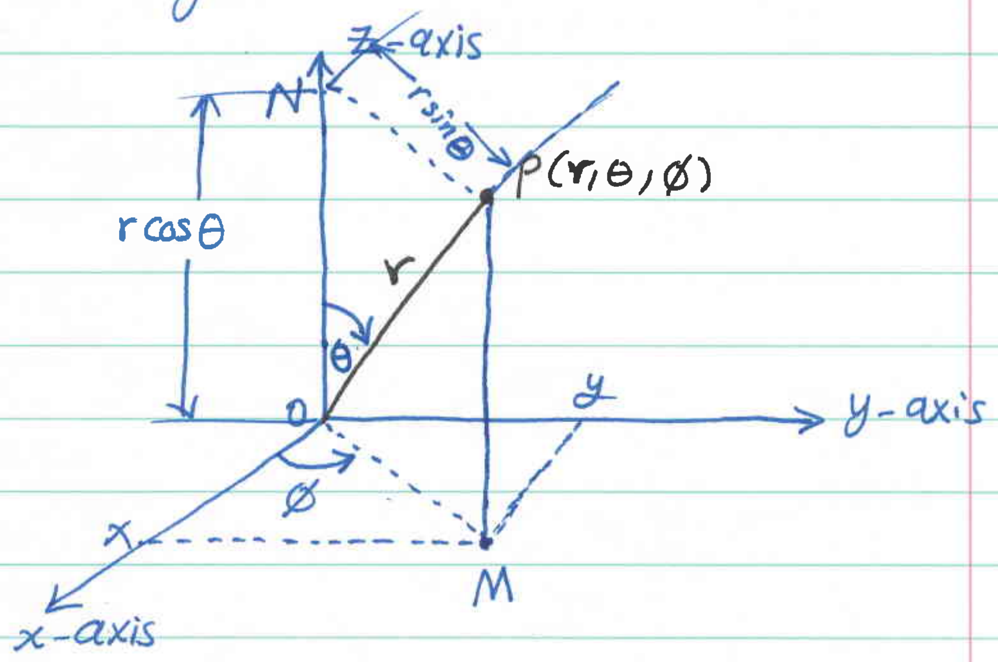
$$\bar{A} = \rho\,\bar{a}_\rho + z\,\bar{a}_z$$

H.W If $\bar{A} = 3\bar{a}_\rho + 2\bar{a}_\phi + 5\bar{a}_z$ and
$\bar{B} = -2\bar{a}_\rho + 3\bar{a}_\phi - \bar{a}_z$
are given at points $P(3, \pi/6, 5)$ and $Q(4, \pi/3, 3)$, find: $\vec{C} = \bar{A} + \bar{B}$ at Point $S(2, \frac{\pi}{4}, 4)$

Ans: $\bar{C} = 0.707\,\bar{a}_\rho + 3.535\,\bar{a}_\phi + 4\,\bar{a}_z$

Spherical coordinate system

A point P in space in spherical coordinates is uniquely represented in term of r, θ, and ϕ as illustrated in figure



where : r : is the radial distance from O to P

θ : is the angle that r makes with the Positive z axis

ϕ : is the angle between the positive xz and OMPN planes

The Projection of r onto the xy-plane is $OM = r\sin\theta$

From figure

$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

We can deduce that

$$r = \sqrt{x^2+y^2+z^2} \qquad 0 \leq r \leq \infty$$

$$\theta = \cos^{-1}\left[\frac{z}{r}\right] \qquad \theta\ [0 \to \pi]$$

$$\phi = \tan^{-1}\left[y/x\right] \qquad \phi\ [0 \to 2\pi]$$

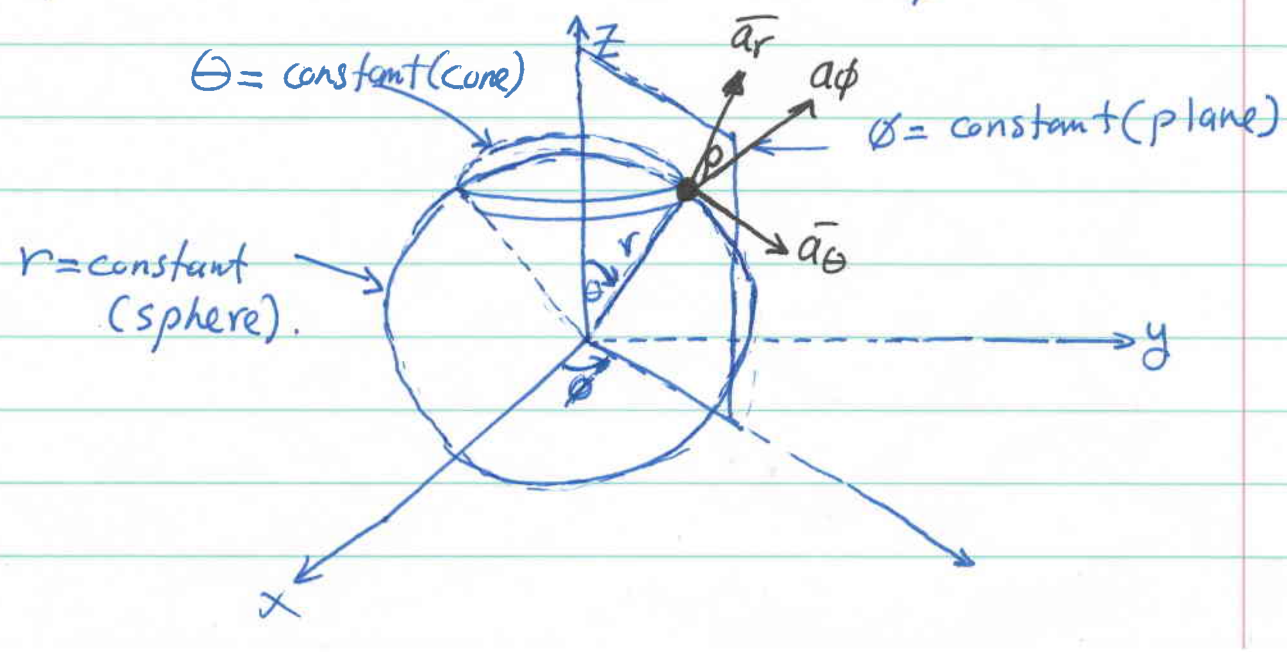
r = constant represent a surface of a sphere of radius r

θ = constant represent a surface of a cone of a parture θ with apex at the origin.

φ = constant represent the plane, hinged on the z-axis.

The tangent planes to these surfaces at Point P are mutually perpendicular.

The unit vector perpendicular to these intersecting planes are $\bar{a}_r$, $\bar{a}_\theta$ and $\bar{a}_\phi$



Important Note :

The vector addition, subtraction and multiplication of any two vectors in spherical coordinates can only be performed if these vectors are given at the intersecting of θ = constant and ϕ = constant.

In other words, the vectors must be defined either at the same point or at a points along the same radial line.

Scalar & Vector Product of Unit Vector

$$\bar{a}_r \cdot \bar{a}_r = 1 \qquad \bar{a}_r \cdot \bar{a}_\theta = 0$$

$$\bar{a}_\theta \cdot \bar{a}_\theta = 1 \qquad \bar{a}_\theta \cdot \bar{a}_\phi = 0$$

$$\bar{a}_\phi \cdot \bar{a}_\phi = 1 \qquad \bar{a}_\phi \cdot \bar{a}_r = 0$$

$$\bar{a}_r \times \bar{a}_\theta = \bar{a}_\phi$$

$$\bar{a}_\theta \times \bar{a}_\phi = \bar{a}_r$$

$$\bar{a}_\phi \times \bar{a}_r = \bar{a}_\theta$$

Transformation of Unit vectors

$$\begin{bmatrix} \bar{a}_r \\ \bar{a}_\theta \\ \bar{a}_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} \bar{a}_x \\ \bar{a}_y \\ \bar{a}_z \end{bmatrix}$$

Transformation of a Vector

If $\bar{A} = A_r \bar{a}_r + A_\theta \bar{a}_\theta + A_\phi \bar{a}_\phi$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

Likewise

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

Example: A vector $\bar{F} = 3x\bar{a}_x + 0.5y^2\bar{a}_y + 0.25x^2y^2\bar{a}_z$ is given at a point $P(3,4,12)$ in the rectangular coordinate system. Express this vector in the spherical coordinate system.

Solution: the vector F at point $P(3,4,12)$ is

$$\bar{F} = 9\bar{a}_x + 8\bar{a}_y + 36\bar{a}_z$$

$$\Rightarrow \quad \phi = \tan^{-1}\left[\frac{4}{3}\right] = 53.13^\circ$$

$$\theta = \cos^{-1}\left[\frac{12}{13}\right] = 22.62^\circ$$

Substituting the values in the Matrix form above.

getting $\Rightarrow$ $F_r = 37.77$ $\quad F_\theta = -2.95$ $\quad F_\phi = -2.40$

or $\bar{F} = 37.77\bar{a}_r - 2.9\bar{a}_\theta - 2.4\bar{a}_\phi$ at $P(13, 22.62^\circ, 53.13^\circ)$.

Differential Elements

* A rectangular differential volume is formed when a point (x, y, z) moved by an incremental distance dx, dy & dz in each of the three directions.

* A section of differential size in cylindrical volume is formed when point (ρ, ϕ, z) move by an incremental distance $d\rho$, $\rho d\phi$ & dz in each of the three direction.

* The differential size spherical volume element is formed by considering incremental distance dr, $r d\theta$ & $r\sin\theta\, d\phi$ from coordinate (r, θ, ϕ).

Cartesian	Cylindrical	spherical.
$\overline{dl} = dx\,\bar{a}_x + dy\,\bar{a}_y + dz\,\bar{a}_z$	$\overline{dl} = d\rho\,\bar{a}_\rho + \rho\, d\phi\,\bar{a}_\phi + dz\,\bar{a}_z$	$\overline{dl} = dr\,\bar{a}_r + r\,d\theta\,\bar{a}_\theta + r\sin\theta\, d\phi\,\bar{a}_\phi$
$d\bar{S}_x = dy\,dz$ $d\bar{S}_y = dx\,dz$ $d\bar{S}_z = dx\,dy$	$d\bar{S}_\rho = \rho\, d\phi\, dz$ $dS_\phi = d\rho\, dz$ $dS_z = \rho\, d\rho\, d\phi$	$dS_r = r^2 \sin\theta\, d\theta\, d\phi$ $dS_\theta = r\sin\theta\, dr\, d\phi$ $dS_\phi = r\,dr\,d\theta$
$dv = dx\,dy\,dz$	$dv = \rho\, d\rho\, d\phi\, dz$	$dv = r^2 \sin\theta\, dr\, d\theta\, d\phi$

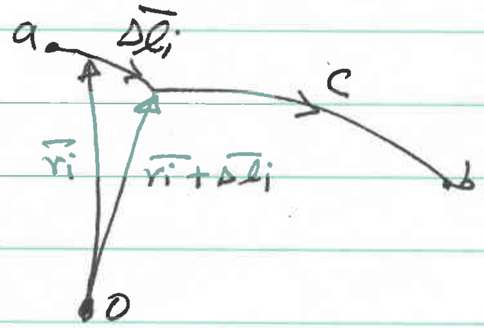
Line, Surface, Volume integrals

We often express the basic laws of electromagnetic fields in terms of integrals of field quantities over various lines, surfaces, and volumes in a region.

let as consider f as a sclar field quantity then $\bar{F}$ is the vector field quantity.

The line integral: The line integral of f along c is then define in the limit of the sum as

$$\int_c f\, d\bar{l} = \lim_{\substack{n \to \infty \\ \Delta l_i \to 0}} \sum_{i=1}^{n} f_i \Delta l_i$$



Where f_i is the value of the scalar function f within the length segment $\overline{\Delta l_i}$. It is evident that this integral is a vector.

The scalar line integral for a vector field $\bar{F}$ is

$$\int_c \bar{F} \cdot d\bar{l} = \lim_{\substack{n \to \infty \\ \Delta l_i \to 0}} \sum_{i=1}^{n} \bar{F}_i \cdot \overline{\Delta l_i}$$

The vector line integral for a vecter field $\bar{F}$ is

$$\int_c \bar{F} \times d\bar{l} = \lim_{\substack{n \to \infty \\ \Delta l_i \to 0}} \sum_{i=1}^{n} \bar{F}_i \times \overline{\Delta l}$$

Example : If $\bar{A} = (4x+9y)\bar{a_x} - 14yz\,\bar{a_y} + 8x^2z\,\bar{a_z}$

evaluate $\int_c \bar{A}\cdot\bar{dl}$ from $p(0,0,0)$ to $Q(1,1,1)$

along the following paths :-

(a) $x = t$, $y = t^2$ and $z = t^3$

(b) The straight lines from $(0,0,0)$ to $(1,0,0)$ then to $(1,1,0)$ and finally to $(1,1,1)$

(c) The straight line joining $p(0,0,0)$ to $Q(1,1,1)$.

Solution :

(a) $\bar{A}\cdot\bar{dl} = (4x+9y)dx - 14yz\,dy + 8x^2z\,dz$

$x = t \implies dx = dt$

$y = t^2 \implies dy = 2t\,dt$

$z = t^3 \implies dz = 3t^2dt$

$$\therefore \int_c \bar{A}\cdot\bar{dl} = \int_{t=0}^{1} [4t + 9t^2 - 28t^6 + 24t^7]\,dt = 4$$

(b) There are three regular Path.

path C_1 : $y=0$, $dy=0$

$z=0$, $dz=0$

$0 \le x \le 1$.

$$\int_{c_1} \bar{A}\cdot\bar{dl} = \int_0^1 4x\,dx = 2$$

path C_2 : $x=1$, $dx=0$,

$z=0$, $dz=0$ and $0 \le y \le 1$

$$\int_{c_2} \bar{A}\cdot\bar{dl} = 0$$

path C_3: $x=1,\ dx=0$

$y=1,\ dy=0$

$0 \leqslant z \leqslant 1$

$$\int_{C_3} \bar{A}\cdot\bar{dl} = \int_0^1 8z\,dz = 4$$

Thus, the line integral from P to Q along the three paths is

$$\int_C \bar{A}\cdot\bar{dl} = \int_{C_1} \bar{A}\cdot\bar{dl} + \int_{C_2} \bar{A}\cdot\bar{dl} + \int_{C_3} \bar{A}\cdot\bar{dl}$$

$$= 2 + 0 + 4 = 6$$

(c) Along the path from P to Q, we have

$$\left.\begin{array}{l} 0 \leqslant x \leqslant 1 \\ 0 \leqslant y \leqslant 1 \\ 0 \leqslant z \leqslant 1 \end{array}\right\} \Rightarrow \left.\begin{array}{c} y = x \\ \text{and} \\ z = x \end{array}\right\} \Rightarrow \begin{array}{l} dy = dx \\ dz = dx \end{array}$$

Therefore,

$$\int_C \bar{A}\cdot\bar{dl} = \int_0^1 (13x - 14x^2 + 8x^3)\,dx$$

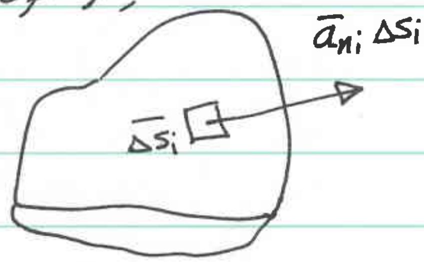
$$= 3.833$$

Note:

The path of integration can be around a closed curve, such a closed path is usually denoted by writing the integral sign as $\oint$.

The surface integral:

To define the surface integral of f, we multiply f by each surface element Δs_i and sum it for all n elements of S in the limit $\overline{\Delta s} \to 0$ as $n \to \infty$.



$$\int_S f \, \overline{ds} = \lim_{\substack{n \to \infty \\ \Delta s_i \to 0}} \sum_{i=1}^{n} f_i \, \overline{\Delta s_i}$$

f_i is the value of the scalar function f over the elemental surface $\overline{\Delta s_i}$. It is clear that the integral is a vector in above eq.

The scalar surface integral:

$$\int_S \bar{F} \cdot \overline{ds} = \lim_{\substack{n \to \infty \\ \overline{\Delta s_i} \to 0}} \sum_{i=1}^{n} \bar{F} \cdot \overline{\Delta s_i}$$

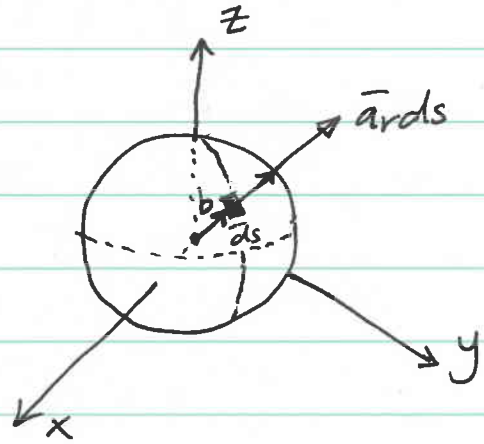
The vector surface integral:

$$\int_S \bar{F} \times \overline{ds} = \lim_{\substack{n \to \infty \\ \overline{\Delta s_i} \to 0}} \sum_{i=1}^{n} \bar{F} \times \overline{\Delta s_i}$$

Example:- Show that over the closed surface of sphere of radius b, $\oint \overline{ds} = 0$

Solution:

$$\oint_s \overline{ds} = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \bar{a}_r b^2 \sin\theta \, d\theta \, d\phi$$



$$\bar{a}_r = \sin\theta\cos\phi \, \bar{a}_x + \sin\theta\sin\phi \, \bar{a}_y + \cos\theta \, \bar{a}_z$$

$$\oint_s \overline{ds} = \bar{a}_x b^2 \int_0^{\pi} \sin^2\theta \, d\theta \int_0^{2\pi} \cos\phi \, d\phi +$$

$$+ \bar{a}_y b^2 \int_0^{\pi} \sin^2\theta \, d\theta \int_0^{2\pi} \sin\phi \, d\phi + \bar{a}_z b^2 \int_0^{\pi} \sin\theta\cos\theta \, d\theta \int_0^{2\pi} d\phi$$

$$= 0$$

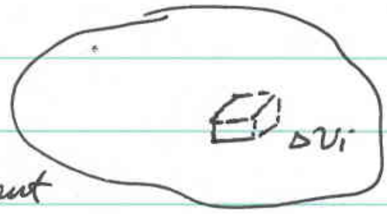
H.w: Evaluate $\oint \bar{r} \cdot \overline{ds}$ over the closed surface of the cube bounded by $0 \leq x \leq 1$, $0 \leq y \leq 1$, and $0 \leq z \leq 1$

wher $\bar{r}$ is the position vector of any point on the surface of the cube.

ans: $\oint \bar{r} \cdot \overline{ds} = 3$

The volume integral :

To define a volume integral, we divide a given volume into n small volume elements as shown in figure. Each element $\Delta v \longrightarrow 0$ as $n \longrightarrow \infty$



$$\therefore \int_v f \, dv = \lim_{\substack{n \to \infty \\ \Delta v_i \to 0}} \sum_{i=1}^{n} f_i \, \Delta v_i$$

as scalar volume integral.

Likewise, we define the volume integral of a vector field $\bar{F}$ as

$$\int_v \bar{F} \, dv = \lim_{\substack{n \to \infty \\ \Delta v_i \to 0}} \sum_{i=1}^{n} \bar{F}_i \, \Delta v_i$$

Example: The electron density distribution within a spherical volume with radius of 2 meters is given as $n_e = (1000/r) \cos(\phi/4)$ electrons/meter^3. Find the charge enclosed if the charge on an electron is -1.6×10^{-19} colomb.

Solution:

let N = the no. of Electrons in the region bounded by a sphere of 2-meter radiu[s]

$$N = \int_v n_e \, dv = \int_v \frac{1000}{r} \cos(\phi/4) \, dv$$

$$= \int_0^2 \frac{1000}{r} r^2 dr \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} \cos(\phi/4) \, d\phi$$

$$= 16{,}000 \text{ electrons} \Rightarrow Q = 16{,}000(-1.6 \times 10^{-19}) = -2.56 \times 10^{-15} \text{ C}$$