3 CONTINUOUS DISTRIBUTIONS

In this section several parametric families of univariate probability density functions are presented. Sketches of some are included; the mean and variance (when they exist) of each are given.

3.1 Uniform or Rectangular Distribution

A very simple distribution for a continuous random variable is the uniform distribution. It is particularly useful in theoretical statistics because it is convenient to deal with mathematically.

Definition 10 Uniform distribution If the probability density function of a random variable X is given by

$$f_X(x) = f_X(x; a, b) = \frac{1}{b-a} I_{[a,b]}(x), \qquad (21)$$

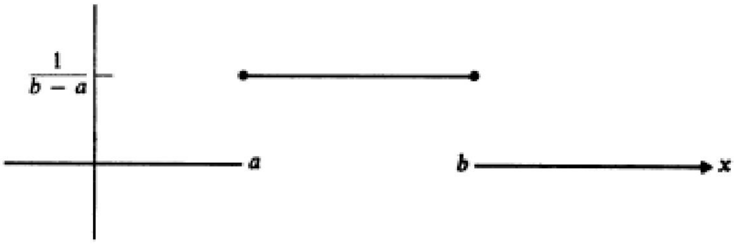

FIGURE 8
Uniform probability density.

where the parameters a and b satisfy $-\infty < a < b < \infty$, then the random variable X is defined to be *uniformly* distributed over the interval $[a, b]$, and the distribution given by Eq. (21) is called a *uniform distribution*.

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Theorem 12 If X is uniformly distributed over $[a, b]$, then

$$\mathscr{E}[X] = \frac{a+b}{2}, \quad \text{var}[X] = \frac{(b-a)^2}{12}, \quad \text{and} \quad m_X(t) = \frac{e^{bt} - e^{at}}{(b-a)t}. \qquad (22)$$

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PROOF

$$\mathscr{E}[X] = \int_a^b x \frac{1}{b-a} dx = \frac{b^2 - a^2}{2(b-a)} = \frac{a+b}{2}.$$

$$\text{var}[X] = \mathscr{E}[X^2] - (\mathscr{E}[X])^2 = \int_a^b x^2 \frac{1}{b-a} dx - \left(\frac{a+b}{2}\right)^2$$

$$= \frac{b^3 - a^3}{3(b-a)} - \frac{(a+b)^2}{4} = \frac{(b-a)^2}{12}.$$

$$m_X(t) = \mathscr{E}[e^{tX}] = \int_a^b e^{tx} \frac{1}{b-a} dx = \frac{e^{bt} - e^{at}}{(b-a)t}.$$ ////

The uniform distribution gets its name from the fact that its density is uniform, or constant, over the interval $[a, b]$. It is also called the *rectangular* distribution—the shape of the density is rectangular.

The cumulative distribution function of a uniform random variable is given by

$$F_X(x) = \left(\frac{x-a}{b-a}\right) I_{[a,b]}(x) + I_{(b,\infty)}(x). \tag{23}$$

3.2 Normal Distribution

A great many of the techniques used in applied statistics are based upon the normal distribution; it will frequently appear in the remainder of this book.

Definition 11 Normal distribution A random variable X is defined to be *normally* distributed if its density is given by

$$f_X(x) = f_X(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}, \tag{24}$$

where the parameters μ and σ satisfy $-\infty < \mu < \infty$ and $\sigma > 0$. Any distribution defined by a density function given in Eq. (24) is called a *normal distribution*. ////

We have used the symbols μ and σ^2 to represent the parameters because these parameters turn out, as we shall see, to be the mean and variance, respectively, of the distribution.

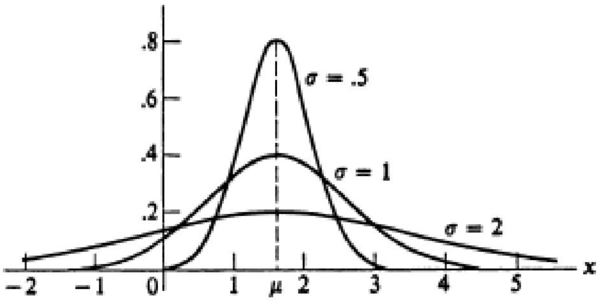

FIGURE 9
Normal densities.

One can readily check that the mode of a normal density occurs at $x = \mu$ and inflection points occur at $\mu - \sigma$ and $\mu + \sigma$. (See Fig. 9.) Since the normal distribution occurs so frequently in later chapters, special notation is introduced for it. If random variable X is normally distributed with mean μ and variance σ^2, we will write $X \sim N(\mu, \sigma^2)$. We will also use the notation $\phi_{\mu,\sigma^2}(x)$ for the density of $X \sim N(\mu, \sigma^2)$ and $\Phi_{\mu,\sigma^2}(x)$ for the cumulative distribution function.

If the normal random variable has mean 0 and variance 1, it is called a *standard* or *normalized* normal random variable. For a standard normal random variable the subscripts of the density and distribution function notations are dropped; that is,

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \quad \text{and} \quad \Phi(x) = \int_{-\infty}^{x} \phi(u)\, du. \tag{25}$$

Since $\phi_{\mu,\sigma^2}(x)$ is given to be a density function, it is implied that

$$\int_{-\infty}^{\infty} \phi_{\mu,\sigma^2}(x)\, dx = 1,$$

Theorem 13 If X is a normal random variable,

$$\mathscr{E}[X] = \mu, \qquad \text{var}[X] = \sigma^2, \qquad \text{and} \qquad m_X(t) = e^{\mu t + \sigma^2 t^2/2}. \tag{26}$$

PROOF

$$\begin{aligned} m_X(t) &= \mathscr{E}[e^{tX}] = e^{t\mu}\mathscr{E}[e^{t(X-\mu)}] \\ &= e^{t\mu} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{t(x-\mu)} e^{-(1/2\sigma^2)(x-\mu)^2}\, dx \\ &= e^{t\mu} \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} e^{-(1/2\sigma^2)[(x-\mu)^2 - 2\sigma^2 t(x-\mu)]}\, dx. \end{aligned}$$

If we complete the square inside the bracket, it becomes

$$\begin{aligned} (x-\mu)^2 - 2\sigma^2 t(x-\mu) &= (x-\mu)^2 - 2\sigma^2 t(x-\mu) + \sigma^4 t^2 - \sigma^4 t^2 \\ &= (x - \mu - \sigma^2 t)^2 - \sigma^4 t^2, \end{aligned}$$