$$m_X(t) = e^{t\mu} e^{\sigma^2 t^2/2} \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-(x-\mu-\sigma^2 t)^2/2\sigma^2}\, dx.$$

The integral together with the factor $1/\sqrt{2\pi}\sigma$ is necessarily 1 since it is the area under a normal distribution with mean $\mu + \sigma^2 t$ and variance σ^2. Hence,

$$m_X(t) = e^{\mu t + \sigma^2 t^2/2}.$$

On differentiating $m_X(t)$ twice and substituting $t = 0$, we find

$$\mathscr{E}[X] = m'_X(0) = \mu$$

and

$$\text{var}\,[X] = \mathscr{E}[X^2] - (\mathscr{E}[X])^2 = m''_X(0) - \mu^2 = \sigma^2,$$

thus justifying our use of the symbols μ and σ^2 for the parameters. ////

Since the indefinite integral of $\phi_{\mu,\sigma^2}(x)$ does not have a simple functional form, one can only exhibit the cumulative distribution function as

$$\Phi_{\mu,\sigma^2}(x) = \int_{-\infty}^{x} \phi_{\mu,\sigma^2}(u)\, du. \qquad (27)$$

Theorem 14 If $X \sim N(\mu, \sigma^2)$, then

$$P[a < X < b] = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right). \qquad (28)$$

Remark $\Phi(x) = 1 - \Phi(-x)$.

EXAMPLE 13 Suppose that an instructor assumes that a student's final score is the value of a normally distributed random variable. If the instructor decides to award a grade of A to those students whose score exceeds $\mu + \sigma$, a B to those students whose score falls between μ and $\mu + \sigma$, a C if a score falls between $\mu - \sigma$ and μ, a D if a score falls between $\mu - 2\sigma$ and $\mu - \sigma$, and an F if the score falls below $\mu - 2\sigma$, then the proportions of each grade given can be calculated. For example, since

$$P[X > \mu + \sigma] = 1 - P[X < \mu + \sigma] = 1 - \Phi\left(\frac{\mu + \sigma - \mu}{\sigma}\right)$$
$$= 1 - \Phi(1) \approx .1587,$$

one would expect 15.87 percent of the students to receive A's. ////

EXAMPLE 14 Suppose that the diameters of shafts manufactured by a certain machine are normal random variables with mean 10 centimeters and standard deviation .1 centimeter. If for a given application the shaft must meet the requirement that its diameter fall between 9.9 and 10.2 centimeters, what proportion of the shafts made by this machine will meet the requirement?

$$P[9.9 < X < 10.2] = \Phi\left(\frac{10.2 - 10}{.1}\right) - \Phi\left(\frac{9.9 - 10}{.1}\right)$$
$$= \Phi(2) - \Phi(-1) \approx .9772 - .1587 = .8185. \qquad ////$$

3.3 Exponential and Gamma Distributions

Two other families of distributions that play important roles in statistics are the (negative) exponential and gamma distributions, which are defined in this subsection. The reason that the two are considered together is twofold; first, the exponential is a special case of the gamma, and, second, the sum of independent identically distributed exponential random variables is gamma-distributed, as we shall see in Chap. V.

Definition 12 Exponential distribution If a random variable X has a density given by

$$f_X(x; \lambda) = \lambda e^{-\lambda x} I_{[0, \infty)}(x), \qquad (29)$$

where $\lambda > 0$, then X is defined to have an (negative) *exponential distribution*. ////

Definition 13 Gamma distribution If a random variable X has density given by

$$f_X(x; r, \lambda) = \frac{\lambda}{\Gamma(r)} (\lambda x)^{r-1} e^{-\lambda x} I_{[0, \infty)}(x), \qquad (30)$$

where $r > 0$ and $\lambda > 0$, then X is defined to have a *gamma distribution*. $\Gamma(\cdot)$ is the gamma function and it is discussed in Appendix A. ////

Remark If in the gamma density $r = 1$, the gamma density specializes to the exponential density. ////

Theorem 15 If X has an exponential distribution, then

$$\mathcal{E}[X] = \frac{1}{\lambda}, \qquad \text{var}[X] = \frac{1}{\lambda^2}, \qquad \text{and} \qquad m_X(t) = \frac{\lambda}{\lambda - t} \qquad \text{for} \qquad t < \lambda. \qquad (31)$$

Theorem 16 If X has a gamma distribution with parameters r and λ, then

$$\mathscr{E}[X] = \frac{r}{\lambda}, \qquad \text{var}\,[X] = \frac{r}{\lambda^2}, \qquad \text{and} \qquad m_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^r \qquad \text{for } t < \lambda. \tag{32}$$

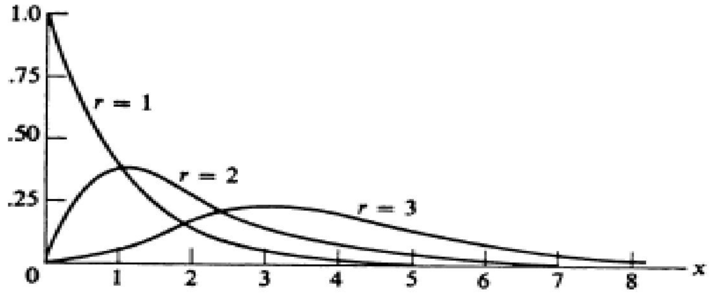

FIGURE 11
Gamma densities ($\lambda = 1$).

PROOF

$$m_X(t) = \mathscr{E}[e^{tX}]$$
$$= \int_0^\infty \frac{\lambda^r}{\Gamma(r)} e^{tx} x^{r-1} e^{-\lambda x}\, dx$$
$$= \left(\frac{\lambda}{\lambda - t}\right)^r \int_0^\infty \frac{(\lambda - t)^r}{\Gamma(r)} x^{r-1} e^{-(\lambda - t)x}\, dx = \left(\frac{\lambda}{\lambda - t}\right)^r.$$
$$m_X'(t) = r\lambda^r(\lambda - t)^{-r-1}$$

and

$$m_X''(t) = r(r+1)\lambda^r(\lambda - t)^{-r-2};$$

hence

$$\mathscr{E}[X] = m_X'(0) = \frac{r}{\lambda}$$

and

$$\text{var}\,[X] = \mathscr{E}[X^2] - (\mathscr{E}[X])^2$$
$$= m_X''(0) - \left(\frac{r}{\lambda}\right)^2 = \frac{r(r+1)}{\lambda^2} - \left(\frac{r}{\lambda}\right)^2 = \frac{r}{\lambda^2}. \qquad ////$$

The exponential distribution has been used as a model for lifetimes of various things. When we introduced the Poisson distribution, we spoke of certain happenings, for example, particle emissions, occurring in time. The length of the time interval between successive happenings can be shown to have an exponential distribution provided that the number of happenings in a fixed