Ch. 3 : Kinetics of Particles

B. Work and Energy:

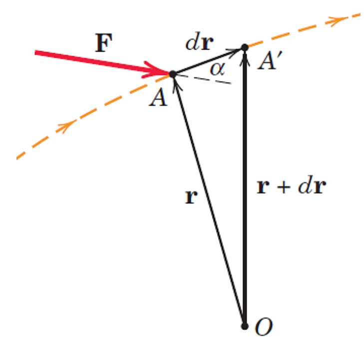
1. Work and Kinetic Energy:

The work-energy method is useful in calculating the change in speed during a displacement of the particle. In this section we will apply work and energy methods to solve plane motion problems involving force, velocity, and displacement

Work (U): Whenever a force ***F*** will do work on a particle only when the particle undergoes *a displacement in the direction of the force.*

Work done = Force × Displacement

The work done by the force **F** during the displacement $d\boldsymbol{r}$ is defined as:



$dU = \mathrm{F} \cdot \mathrm{dr}$

$U = \mathrm{F}\, ds \cos \alpha$

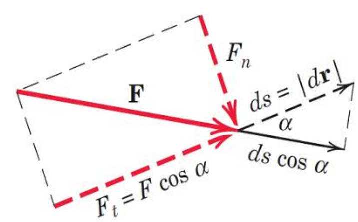
where: $ds = |dr|$

F_t **:** tangential component of force F

F_n **:** normal component of force F

$F_t = F \cos \alpha$ (in the direction of displacement ds and ***dose work***)

$F_n = F \sin \alpha$ **(**normal to the displacement ds and ***does no work***)

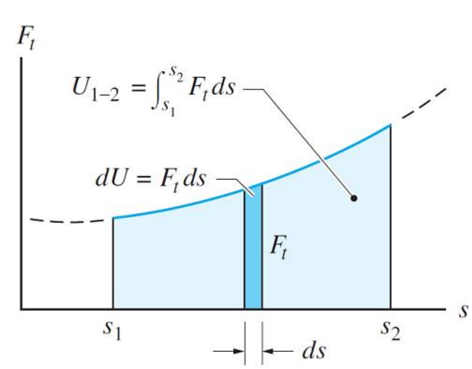


$\therefore \quad dU = F_t\, ds$

$U_{1-2} = \int_{s_1}^{s_2} F_t\, ds$

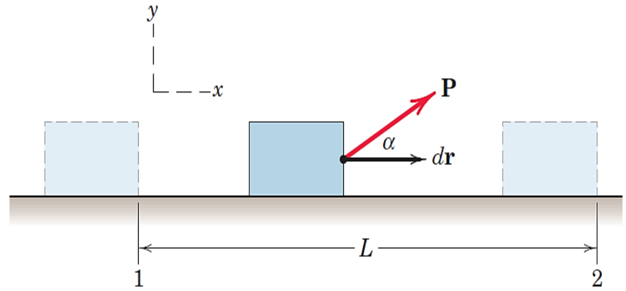
The unit of work U is ***Joules (J)***

or **N · m , lb · ft**



Calculations of Work:

a- Work of Constant External Force (P)

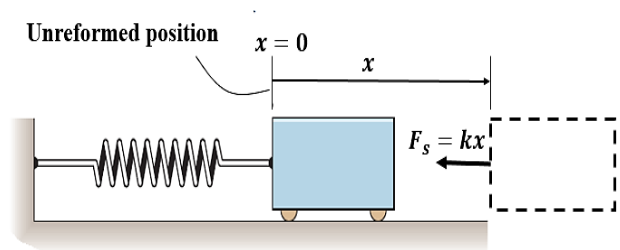


$$U_P = U_{1-2} = \int_{x_1}^{x_2} P \cos\alpha \, dx$$

$$U_P = P \cos\alpha (x_2 - x_1)$$

$$\boldsymbol{U_P = PL \cos\alpha} \qquad (1)$$

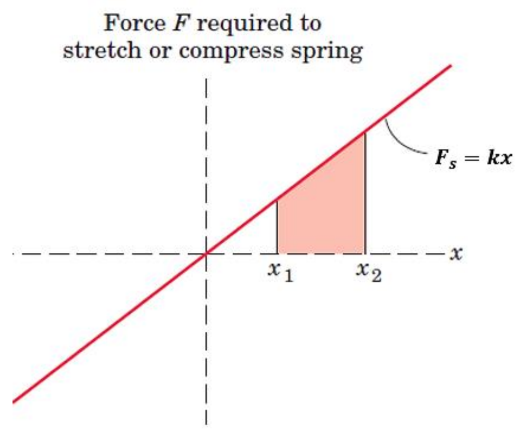
b- Work of Spring Force (F_s)



$$U_{F_s} = -\int_{x_1}^{x_2} F_s \, dx$$

$$U_{F_s} = -\int_{x_1}^{x_2} kx \, dx$$

$$\boldsymbol{U_{F_s} = \frac{1}{2} k(x_1^2 - x_2^2)} \qquad \mathbf{(2)}$$



Where:

$\boldsymbol{k}$: is the spring stiffness (N/m) or (lb/ft)

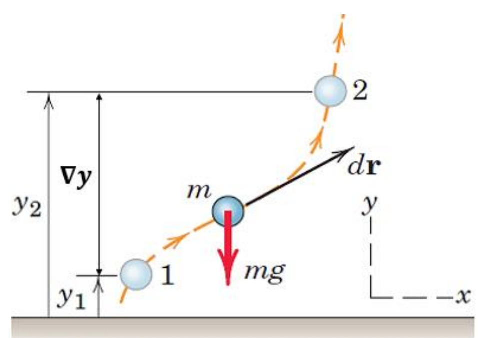
$\boldsymbol{x}$: is the stretch or compression of the spring (m, ft)

c- Work of Weight (W)

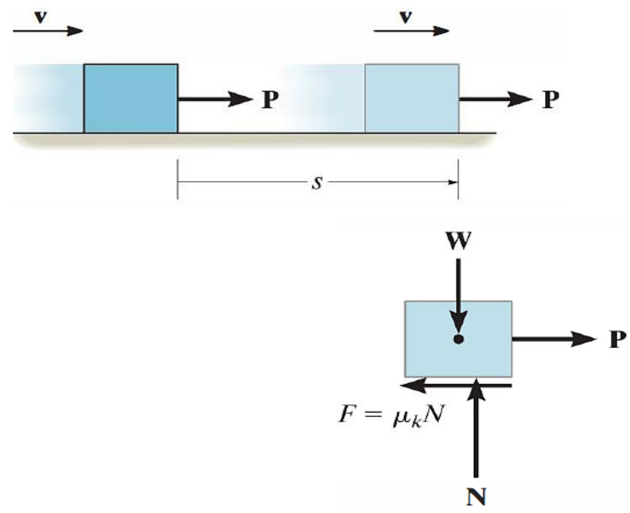
$$U_W = -mg(y_2 - y_1)$$

$$\boldsymbol{U_W = -W\Delta y} \qquad (3)$$

The U_W is positive (+) when the weight moves downwards, and negative (-) when the weight moves upward.



d- Work of Friction Caused by Sliding.



$$U_F = F\,S = (\mu_k N)\,S$$

F: friction force

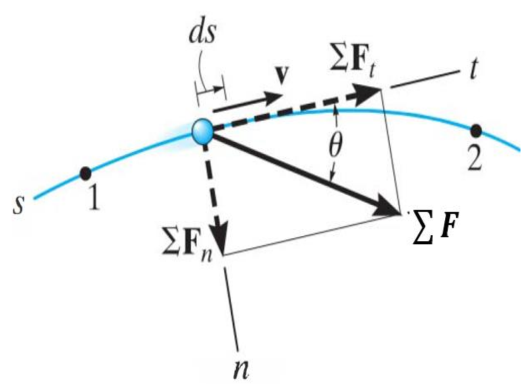
N: normal (reaction) force

μ_k : The coefficient of kinetic friction

The principle of Work and Kinetic Energy:

$\sum F_n$ does no work (always normal to the path s)

$\sum F_t$ done work (always in the direction of path s)



Newton's second law ; $\sum F_t = ma_t$

From kinematics: $a_t ds = v dv$

$$\therefore \quad \frac{\sum F_t}{m} ds = v dv$$

$$\sum \int_1^2 F_t\, ds = m \int_{v_1}^{v_2} v\, dv = \frac{1}{2} m(v_2^2 - v_1^2)$$

The ***kinetic energy T*** of the particle is defined as : $\boldsymbol{T = \frac{1}{2} m v^2}$ (4)

$$\sum \int_1^2 F_t\, ds = \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2$$

$$\sum U_{1-2} = T_2 - T_1 = \Delta T$$

$$\therefore \quad \boldsymbol{T_1 + \sum U_{1-2} = T_2}$$ ***(principle of work and kinetic energy)*** (5)

T_1 : initial kinetic energy (Joule)

T_2 : final kinetic energy (Joule)

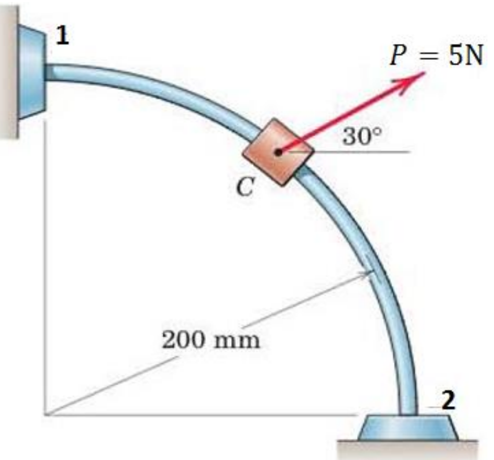
$\sum U_{1-2}$: the total work done by all forces acting on a particle as it moves from point 1 to point 2

Notes:

1- Equation (5) is used to solve the problems that involve the velocity $\boldsymbol{v}$, force **F** , and displacement ***s.***

2- Work is *positive* when the force component is in the *same sense of direction* as its displacement, otherwise it is negative.

Ex. (1): The 0.5kg collar C slides with negligible friction on the fixed rod in the vertical plane. If the collar starts from rest at 1 under the action of the constant 5 N force, calculate its velocity v as it hits the stop at 2. Neglect the small dimensions of the collar.



Sol.:

The collar starts from rest at *A :* $v_1 = 0$, $T_1 = \frac{1}{2}mv_1^2 = 0$

$$T_2 = \frac{1}{2}mv_2^2 = 0.25v_2^2$$

Work of force 5N ; $U_P = P\ s = 5\cos 30\,(0.2) - 5\sin 30\,(0.2) = 0.366$ J

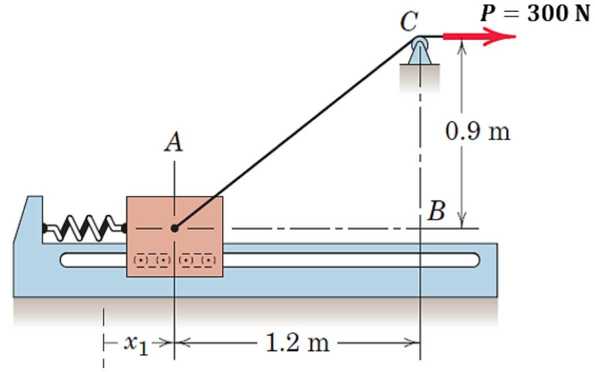
Work of weight W; $U_W = W\ \Delta y = mg(y_2 - y_1) = 0.5(9.81)(0.2 - 0) = 0.981$ J

Total work; $\sum U_{1-2} = U_P + U_W = 1.347$ J

$$\therefore \quad T_1 + \sum U_{1-2} = T_2$$

$0 + 8.943 = 0.4v_2^2$ $\quad\therefore\quad v_2 = 4.73$ m/s **Ans.**

Ex. (2): The 50-kg block at A is mounted on rollers so that it moves along the fixed horizontal rail with negligible friction under the action of the constant $P = 300$ N force in the cable. The block is released from rest at A, with the spring to which it is attached extended an initial amount $x_1 = 0.233$ m. The spring has a stiffness $k = 80$ N/m. Calculate the velocity v of the block as it reaches position B.



Sol.:

The block is released from rest at A: $v_1 = 0$, $T_1 = 0$

$$T_2 = \frac{1}{2} m v_2^2 = 25 v_2^2$$

$$x_1 = 0.233 \text{ m} \quad , \quad x_2 = x_1 + 1.2 = 1.433 \text{ m}$$

Work of spring ; $U_s = \frac{1}{2} k (x_1^2 - x_2^2) = \frac{1}{2}(80)(0.233^2 - 1.433^2) = -80$ J

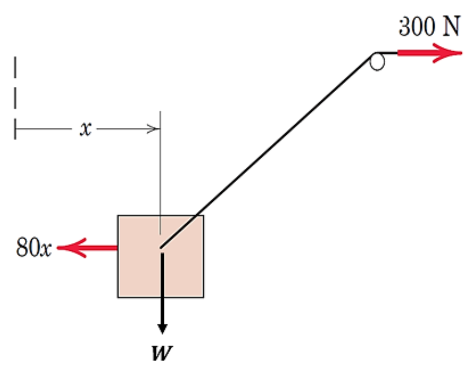
Work of force P ; $U_P = P\, s$

$$s = AC - BC = \sqrt{1.2^2 + 0.9^2} - 0.9 = 0.6 \text{ m}$$

$\therefore$ $U_P = 300(0.6) = 180$ J

Work of weight; $U_W = 0$

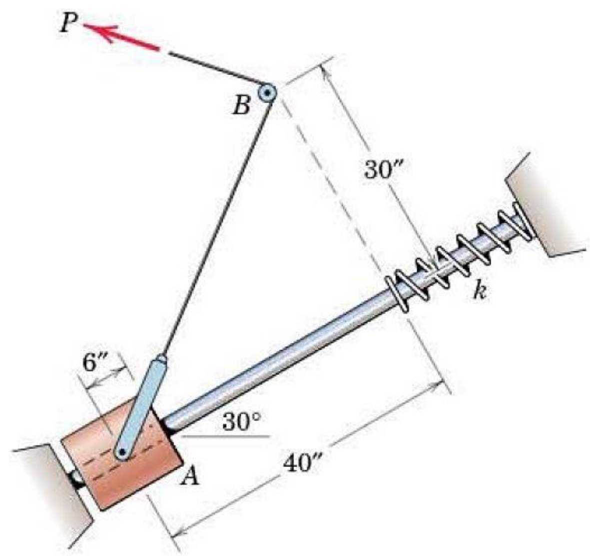
Total work ; $\sum U_{1-2} = U_s + U_P + U_W = 100$ J



$$T_1 + \sum U_{1-2} = T_2$$

$0 + 100 = 25 v_2^2$ $\therefore$ $v = 2$ m/s **Ans.**

Ex. (3): The 32-lb collar A is released from rest in the position shown and slides with negligible friction up the fixed rod inclined 30^o from the horizontal under the action of a constant force $P = 50$ lb applied to the cable. Calculate the required stiffness k of the spring so that its maximum deflection equals 6 in. the position of the small pulley at B is fixed



Sol.:

For collar ; $\sum U_{1-2} = \Delta T = T_2 - T_1$

from rest $v_1 = 0$, $T_1 = 0$

since maximum deflection, the collar is stopped a ; v_2 , $T_2 = 0$

$$U_P = P\,s \quad , \quad s = AB - BC = \sqrt{\left(\frac{40}{12}\right)^2 + \left(\frac{30}{12}\right)^2} - \frac{30}{12} = 1.67 \text{ ft}$$

$\therefore \quad U_P = 50(1.67) = 83.33 \text{ lb} \cdot \text{ft}$

$$U_W = -Wh = -30\left(\frac{40\sin 30}{12}\right) = -50 \text{ lb} \cdot \text{ft}$$

Work of spring ; $U_s = \frac{1}{2}k(s_1^2 - s_2^2) = \frac{1}{2}k\left(0 - \left(\frac{6}{12}\right)^2\right) = -0.125\,k$

$\therefore \sum U_{1-2} = T_2 - T_1 = 0$

$U_P + U_W + U_s = 0$

$83.33 - 50 - 0.125\,k = 0$

$\therefore \quad k = 266.64 \approx 267 \text{ lb/ft}$ **Ans**

2. Conservative Forces and Potential Energy:

Conservative Forces : If the work of a force is independent of the path and depends only on the force's initial and final positions, then we can classify this force as a *conservative force.*

The Conservative Forces are: (**1**) weight ($W = mg$) and (**2**) spring force ($F_s = ks$)

Potential Energy:

When energy comes from the position of the particle, measured from a fixed datum or reference plane, it is called *potential energy*.

The potential energy $\boldsymbol{V}$ is a measure the amount of work done by conservative forces (weight, spring force).

a- Gravitational Potential Energy (V_g).

if y is positive upward, the gravitational potential energy of the particle of weight W is

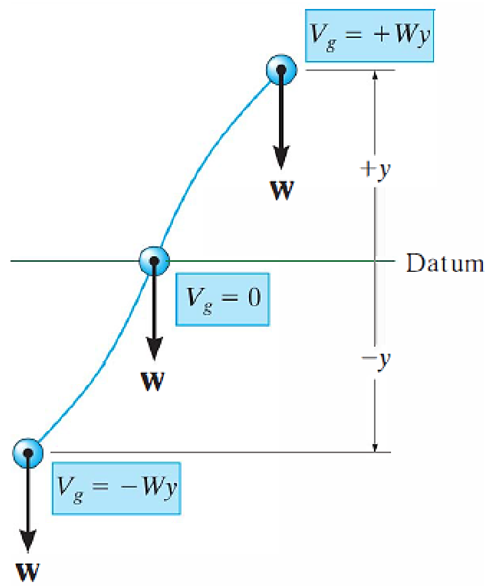
$$\boldsymbol{V_g = Wy}$$

At the datum ($y = 0$) $\quad \boldsymbol{V_g = 0}$

Under the datum $\quad \boldsymbol{V_g = -Wy}$

The change in potential energy is

$$\boldsymbol{\Delta V_g = W\Delta y}$$



Gravitational potential energy

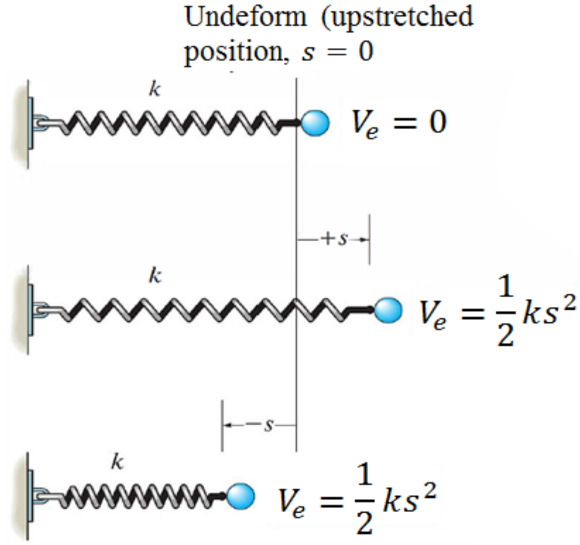
b- Elastic Potential Energy (V_e).

$$V_e = \int_0^s F_s \ ds = \int_0^s ks \ ds$$

$$\boldsymbol{V_e = \frac{1}{2} k s^2}$$

And $\boldsymbol{\Delta V_e = \frac{1}{2} k (\Delta s^2)}$

$$\boldsymbol{\Delta V_e = \frac{1}{2} k (s_2^2 - s_1^2)}$$



Elastic potential energy

The total potential energy is:

$$\boldsymbol{V = V_g + V_e}$$

Work-Energy Equation:

If the particle moves from initial position (1) to the final position (2) then

$$U'_{1-2} = \Delta T + \Delta V$$

Where: $\Delta T = T_2 - T_1$ and $\Delta V = V_2 - V_1$

$$\therefore \qquad \boldsymbol{T_1 + V_1 + U'_{1-2} = T_2 + V_2}$$

$\boldsymbol{U'_{1-2}}$: the work of the external force $\boldsymbol{P}$ (non-conservative force)

Conservative of Energy:

If only conservative forces (weight, spring force) acting on the particle, then

$$P = 0 \ \rightarrow \ U'_{1-2} = 0$$

$$\therefore \qquad \boldsymbol{T_1 + V_1 = T_2 + V_2}$$

{*Law of conservative of energy, or conservative of mechanical energy*}

Or

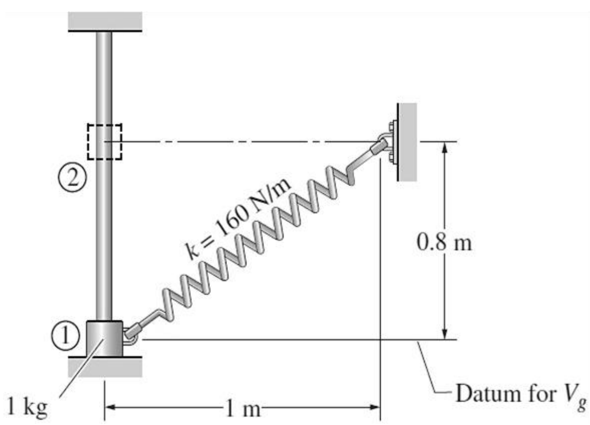
$$\boldsymbol{E_1 = E_2}$$

Where:

$E_1 = T_1 + V_1$: total energy at initial position

$E_2 = T_2 + V_2$: total energy at final Position

Ex. (4): The figure shows a 1-kg collar that slides along the frictionless vertical rod under the actions of gravity and an ideal spring. The spring has a stiffness of 160 N/m, and its free length is 0.9 m. The collar is released from rest in position 1. Determine the speed of the collar in position 2



Sol.:

Select datum at initial position (1)

$$\left(V_g\right)_1 = mgh_1 = 1(9.81)(0) = 0$$

$$\left(V_g\right)_2 = mgh_2 = 1(9.81)(0.8) = 7.84 \text{ J}$$

For spring; $s_1 = l_1 - l_0 = \sqrt{0.8^2 + 1^2} - 0.9 = 0.38 \text{ m}$

$s_2 = l_2 - l_0 = 1 - 0.9 = 0.1 \text{ m}$

$(V_e)_1 = \frac{1}{2}ks_1^2 = \frac{1}{2}(160)(0.38)^2 = 11.552 \text{ J}$, $(V_e)_2 = \frac{1}{2}ks_2^2 = \frac{1}{2}(160)(0.1)^2 = 0.8 \text{ J}$

$\therefore \ V_1 = \left(V_g\right)_1 + (V_e)_1 = 11.552$, $V_2 = \left(V_g\right)_2 + (V_e)_2 = 8.64 \text{ J}$

$T_1 = \frac{1}{2}mv_1^2 = 0$, $T_2 = \frac{1}{2}mv_2^2 = 0.5v_2^2$

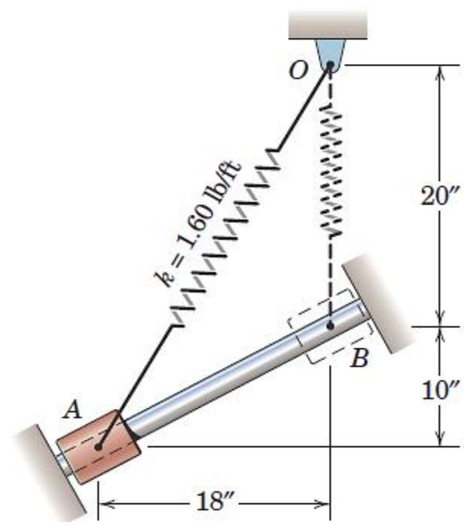
$\sum U'_{1-2} = 0$ (no external force)

The conservation of energy is

$$T_1 + V_1 = T_2 + V_2$$

$$0 + 11.552 = 0.5v_2^2 + 8.64 \qquad \therefore v_2 = 2.41 \text{ m/s} \qquad \textbf{Ans.}$$

Ex. (5): The 2-lb collar is released from rest at A and slides freely up the inclined rod, striking the stop at B with a velocity v. The spring of stiffness $k = 1.6$ lb/ft has an undeform (unstretched) length of 15 in. Calculate v.



Sol.:

$T_A = \frac{1}{2}mv_A^2 = 0$

$T_B = \frac{1}{2}m = \frac{1}{2}\left(\frac{2}{32.2}\right)v^2 = 0.031v^2$

$\left(V_g\right)_A = Wh_A = 2(0) = 0$

$\left(V_g\right)_B = Wh_B = 2(10/12) = 1.667 \text{ lb}\cdot\text{ft}$

For spring; $s_A = l_{OA} - l_0 = \sqrt{18 + 30^2} - 15 = 20$ in

$s_B = l_{OB} - l_0 = 20 - 15 = 5$ in

$(V_e)_A = \frac{1}{2}ks_A^2 = \frac{1}{2}(1.6)\left(\frac{20}{12}\right)^2 = 2.22 \text{ lb}\cdot\text{ft}$, $(V_e)_B = \frac{1}{2}ks_B^2 = \frac{1}{2}(1.6)\left(\frac{5}{12}\right)^2 = 0.1388 \text{ lb}\cdot\text{ft}$

$\therefore$ $V_A = \left(V_g\right)_A + (V_e)_A = 2.22 \text{ lb}\cdot\text{ft}$, $V_B = \left(V_g\right)_B + (V_e)_B = 1.806 \text{ lb}\cdot\text{ft}$

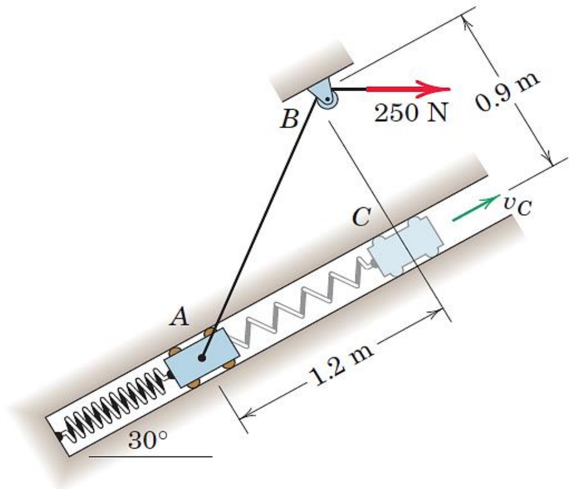
$\sum U'_{A-B} = 0$ (no external force)

The conservation of energy is

$$T_A + V_A = T_B + V_B$$

$$0 + 2.22 = 0.031v^2 + 1.806 \qquad \therefore v = 3.654 \text{ ft/s} \qquad \textbf{Ans.}$$

Ex. (6): The 10 kg slider moves with negligible friction up the inclined guide. The attached spring has a stiffness of 60 N/m and is stretched 0.6 m in position A, where the slider is released from rest. The 250N force is constant and the pulley offers negligible resistance to the motion of the cord. Calculate the velocity v_C of the slider as it passes point *C*.



Sol.:

Select datum at initial position A

$$(V_g)_A = mgh_A = 10(9.81)(0) = 0$$

$$(V_g)_C = mgh_C = 10(9.81)(1.2\sin 30) = 58.9 \text{ J}$$

For spring; $x_A = 0.6$ m , $x_C = x_A + 1.2 = 1.8$ m

$$(V_e)_A = \frac{1}{2}kx_A^2 = \frac{1}{2}(60)(0.6)^2 = 10.8 \text{ J}$$

$$(V_e)_C = \frac{1}{2}kx_C^2 = \frac{1}{2}(60)(1.2)^2 = 92.7 \text{ J}$$

$$\therefore \ V_A = (V_g)_A + (V_e)_A = 10.8 \text{ J} \quad , \qquad V_C = (V_g)_C + (V_e)_C = 151.6 \text{ J}$$

$$\sum U'_{A-C} = Ps = P(AB - AC) = 250\left(\sqrt{1.2^2 + 0.9^2} - 0.9\right) = 150 \text{ J}$$

$$T_A = \frac{1}{2}mv_A^2 = 0 \quad , \qquad T_C = \frac{1}{2}mv_C^2 = 5v_2^2$$

$$T_A + V_A + \sum U'_{A-C} = T_C + V_C$$

$$0 + 10.8 + 150 = 5v_2^2 + 151.6$$

$$\therefore \ v_C = 0.974 \text{ m/s}$$ **Ans.**

H.W:

1- Re-solve examples (1, 2 and 3) by using the Work-Energy Equation.
2- Re-solve examples (4,5 and 6) by using the principle of work and energy.