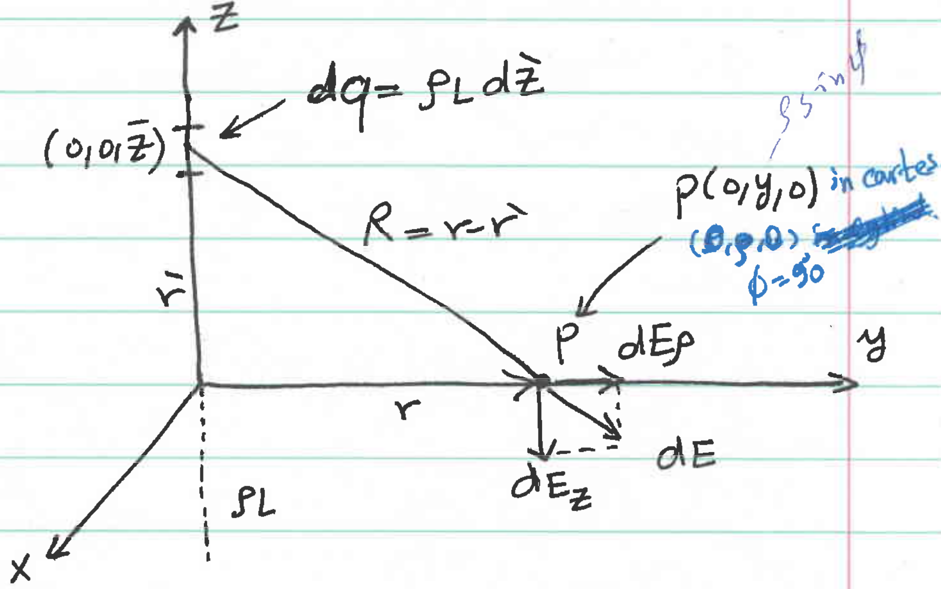
(11)

Field of infinite line charge

Let us asume a straight Line charge extending along the z axis in a cylindrical coordinate system from $-\infty$ to ∞, We desire the electric field intensity E at any and every point resulting from a uniform line charge density ρ_L.



Symmetry should always be considerd first in order to determine two specific factors

(1) With which <u>coordinate</u>, the field does not vary and

(2) Which <u>component</u> of the field are not present.

From testing Symmetry we conclude that:

- the field varies only with ρ.
- We have only an E_ρ component.
- the Contribution of E_z will Canceld.
- No element of charge produces $E_\phi \Rightarrow E_\phi = 0$

(12)

$$dE = \frac{\rho_L dz' (r - r')}{4\pi\epsilon_0 |r - r'|^3}$$

where $r = y a_y$

$\bar{r} = z' a_z$

and $r - \bar{r} = \rho a_\rho - \bar{z} a_z$

Therefore,

$$dE = \frac{\rho_L dz' (\rho \bar{a}_\rho - z' a_z)}{4\pi\epsilon_0 (\rho^2 + z'^2)^{3/2}}$$

Since only the E_ρ component is present, we may simplify:

$$dE_\rho = \frac{\rho_L \rho d\bar{z}}{4\pi\epsilon_0 (\rho^2 + z'^2)^{3/2}}$$

and

$$E_\rho = \int_{-\infty}^{\infty} \frac{\rho_L \rho dz'}{4\pi\epsilon_0 (\rho^2 + z'^2)^{3/2}}$$

By integrating:

$$E_\rho = \frac{\rho_L}{4\pi\epsilon_0} \rho \left(\frac{1}{\rho^2} \times \frac{z'}{\sqrt{\rho^2 + z'^2}} \right)_{-\infty}^{\infty}$$

$$\therefore \boxed{E_\rho = \frac{\rho_L}{2\pi\epsilon_0 \rho}}$$

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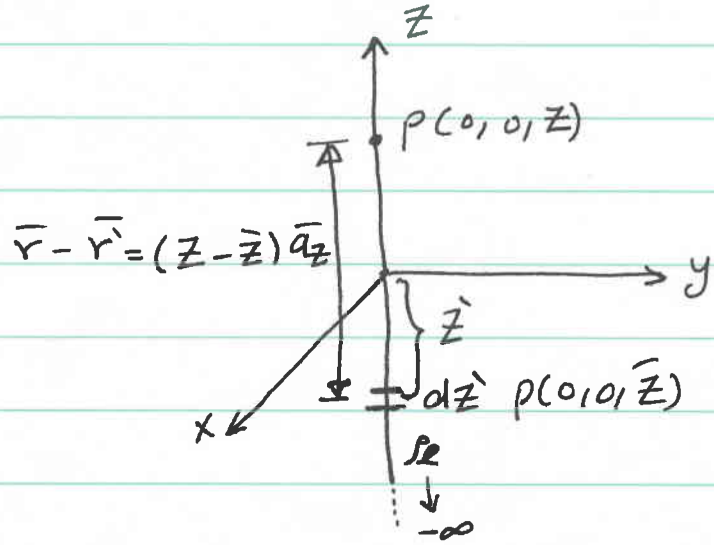
Example:- A semi-infinite line extending from $-\infty$ to 0 along the z-axis carries a uniform charge distribution of $100\ nC/m$. Find the electric field intensity at point $p(0,0,2)$.
If a charge of $1\ \mu C$ is placed at p, calculate the force acting on it.



Solution:

The differential charge element $dq = \rho_\ell dz'$ at $z = z'$ from the origion.

The distance vector from z' to p is $\bar{r}-\bar{r}' = (z - z')\bar{a}_z$
The magnitude of the distance vector $|\bar{r}-\bar{r}'| = z - z'$.

$\therefore$ E at point P is:

$$\bar{E} = \bar{a}_z \frac{\rho_\ell}{4\pi\epsilon_0} \int_{-\infty}^{0} \frac{dz'}{(z-z')^2}$$

$$\therefore \boxed{\bar{E} = \frac{\rho_\ell}{4\pi\epsilon_0 z}\ \bar{a}_z}$$

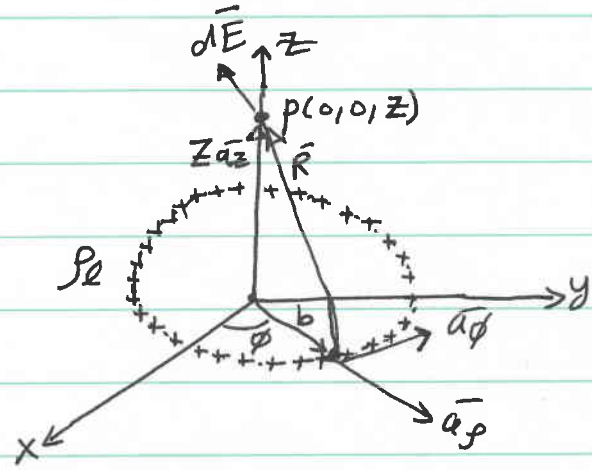
Substituting the values, we obtain

$$\bar{E} = \frac{9\times10^{9} \times 100 \times 10^{-9}}{2} \bar{a}_z = 450\,\bar{a}_z \ \ V/m$$

The force acting on a charge of $1\,\mu C$ at $z = 2m$ is

$$\bar{F} = q\bar{E} = 1\times10^{-6} \times 450\,\bar{a}_z = 450\,\bar{a}_z \ \mu N$$

Example:- The charge is uniformly distribution in the shape of ~~a~~ ring of radius b, as shown in fig. Determine the electric field intensity at any point on the axis of the ring.



Solution:-

$$d\bar{l} = b\,d\phi\,\bar{a}_\phi$$

$$\bar{R} = -b\,\bar{a}_\rho + z\,\bar{a}_z$$

$$\bar{E} = \frac{\rho_l}{4\pi\epsilon_0}\int_0^{2\pi} \frac{b\,d\phi}{[b^2+z^2]^{3/2}}(-b\,\bar{a}_\rho + z\,\bar{a}_z)$$

$$= \frac{\rho_l b}{4\pi\epsilon_0}\frac{1}{[b^2+z^2]^{3/2}}\left[-b\int_0^{2\pi}\bar{a}_\rho\,d\phi' + z\int_0^{2\pi}d\phi'\,\bar{a}_z\right]$$

Because $\bar{a}_\rho = \bar{a}_x\cos\phi' + \bar{a}_y\sin\phi'$

$$\therefore \int_0^{2\pi}\bar{a}_\rho\,d\phi' = \bar{a}_x\int_0^{2\pi}\cos\phi'\,d\phi' + \bar{a}_y\int_0^{2\pi}\sin\phi'\,d\phi' = 0$$

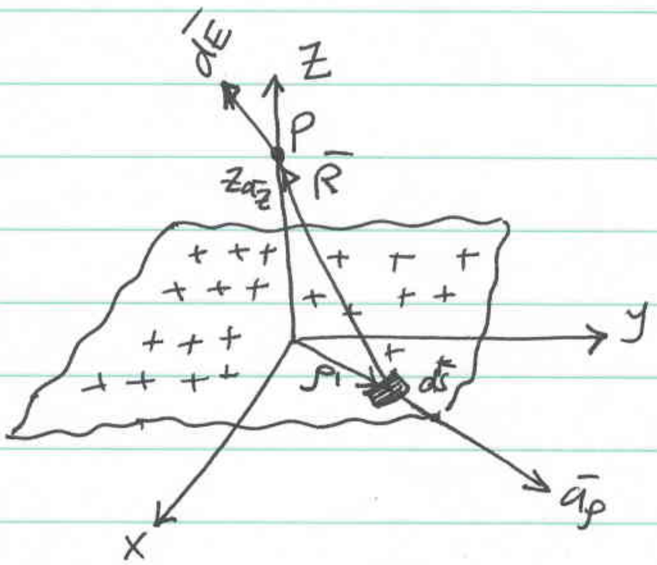
$$\therefore \bar{E} = \frac{\rho_l b z}{2\epsilon_0[b^2+z^2]^{3/2}}\bar{a}_z$$

" Field of Infinite sheet of charges "

If charge is distributed with uniform density ρ_s (c/m^2) over an infinite plane then to find $\bar{E}$ at any point P the clyindrical coordinate system will be used with charge in the $z=0$ plane.

Then

$$\boxed{\bar{E} = \frac{\rho_s}{2\epsilon_0} a_n}$$



$$\bar{R} = -\rho \bar{a}_\rho + z\bar{a}_z \Rightarrow |\bar{R}| = \sqrt{\rho^2+z^2}$$

$$dq = \rho_s\, ds \qquad ds = \rho\, d\rho\, d\phi$$

$$\therefore dq = \rho_s\, \rho\, d\rho\, d\phi$$

$$\therefore d\bar{E} = \frac{\rho_s\, \rho\, d\rho\, d\phi}{4\pi\epsilon_0} \cdot \frac{-\rho\bar{a}_\rho + z\bar{a}_z}{(\rho^2+z^2)^{3/2}}$$

Since, there is a symmetry about the z-axis, it result in cancellation of the radial (ρ) component

$$\therefore \quad \bar{E} = \int_0^{2\pi}\int_0^{\infty} \frac{\rho_s \, \rho \, d\rho \, d\phi}{4\pi\epsilon_0 \, (\rho^2 + z^2)^{3/2}} \, z \, \bar{a}_z$$

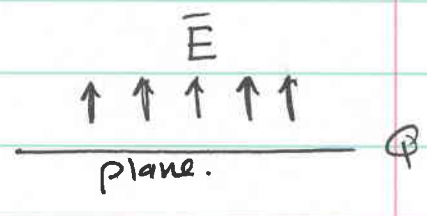
$$= \frac{\rho_s}{4\pi\epsilon_0} \left[\frac{-1}{\sqrt{\rho^2 + z^2}} \right]_0^{\infty} \left[\phi \right]_0^{2\pi}$$

$$\Rightarrow \quad \bar{E} = \frac{\rho_s}{2\epsilon_0} a_z \, .$$

This result is for points above xy-plane, below the xy-plane, the unit vector change to $(-\bar{a}_z)$

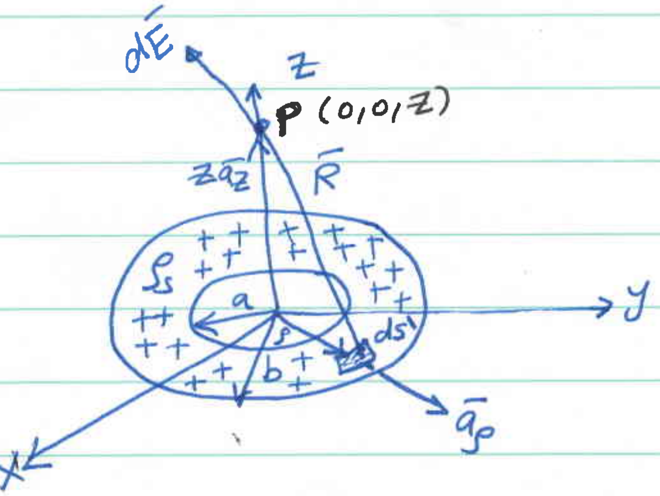
The generalized form may be written using $\bar{a}_n$ (the unit normal vector).

$$\boxed{\bar{E} = \frac{\rho_s}{2\epsilon_0} \bar{a}_n}$$



The electric field is every where normal to the plane of charge and its magnitude is independent of the distance from the plane -

<u>Example</u>: A thin annular disc of inner radius a and outer radius b carries a uniform surface charge density ρ_s. Determine the electric field intensity at any point on the z-axis when $z \geqslant 0$.



solution:

$$dq' = \rho_s \, \rho' d\rho' d\phi'$$

$$\bar{R} = -\bar{\rho} \, \bar{a}_\rho + z \, \bar{a}_z$$

$$|R| = [\rho'^2 + z^2]^{1/2}$$

$\therefore$ The electric field intensity at point $P(0,0,z)$ is

$$\bar{E} = \frac{\rho_s}{4\pi\epsilon_0} \int_a^b \int_0^{2\pi} \frac{\rho' d\rho' d\phi'}{[\rho^2 + z^2]^{3/2}} [-\rho' \bar{a}_\rho + z \bar{a}_z]$$

$$\int_0^{2\pi} \bar{a}_\rho \, d\phi = 0$$

$$\bar{E} = \frac{\rho_s}{4\pi\epsilon_0} \int_a^b \int_0^{2\pi} \frac{\rho' d\rho' d\phi'}{[\rho'^2 + z^2]^{3/2}} z \bar{a}_z$$

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$$= \frac{\rho_s z}{2\epsilon_0}\left[\frac{1}{(a^2+z^2)^{1/2}} - \frac{1}{(b^2+z^2)^{1/2}}\right]\bar{a}_z$$

For an annular disc with very large outer radius $b \to \infty$, the electric field intensity becomes, fig1

$$\bar{E} = \frac{\rho_s z}{2\epsilon_0}\left[\frac{1}{(a^2+z^2)^{1/2}}\right]\bar{a}_z$$

For a solid finite disc of outer radius b, the electric field intensity becomes, fig2

$a=0$

$$\bar{E} = \frac{\rho_s z}{2\epsilon_0}\left[\frac{1}{z} - \frac{1}{(b^2+z^2)^{1/2}}\right]\bar{a}_z$$

Finally the electric field intensity at any point due to an infinite plane of charge, fig3

$a \to 0$

$b \to \infty$ Thus,

$$\boxed{\bar{E} = \frac{\rho_s}{2\epsilon_0}\bar{a}_z}$$

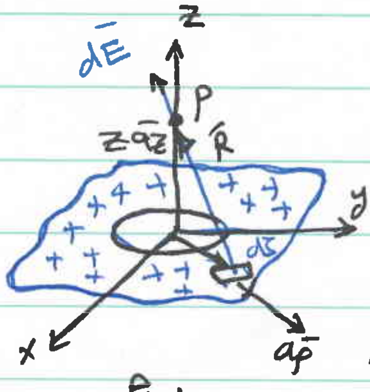


fig1

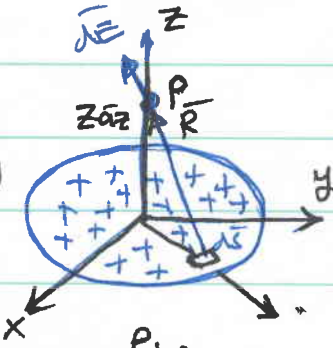


fig2

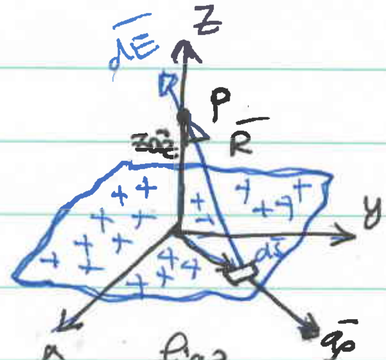
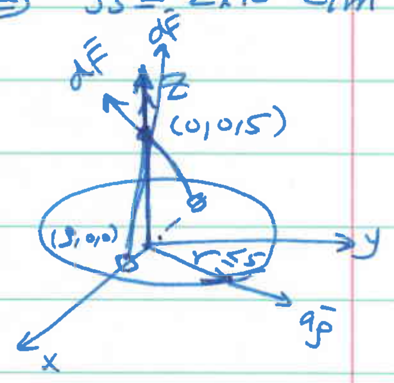


fig3

Example: Find the force on a point charge of $50\ \mu C$ at $(0,0,5)$ m due to a charge of $500\pi\ \mu C$ that is uniformly distributed over the circular disk $r \leq 5$ m, $z = 0$ m

Solution:

$$\rho_s = \frac{Q}{A} = \frac{500\pi \times 10^{-6}}{5^2 \times \pi} \Rightarrow \rho_s = 2\times10^{-5}\ C/m^2$$



In cylindrical coordinates.

$$\bar{R} = -\rho\,\bar{a}_\rho + 5\,\bar{a}_z$$

$$|\bar{R}| = \sqrt{\rho^2+25}$$

$\therefore$

$$d\bar{F} = \frac{(50\times10^{-6})\,(\rho_s\,\rho\,d\rho\,d\phi)}{4\pi\epsilon_0} \cdot \frac{(-\rho\bar{a}_\rho + 5\bar{a}_z)}{(\rho^2+25)^{3/2}}$$

Before Integration, Note that the radial components will cancel and that $\bar{a}_z$ is constant.

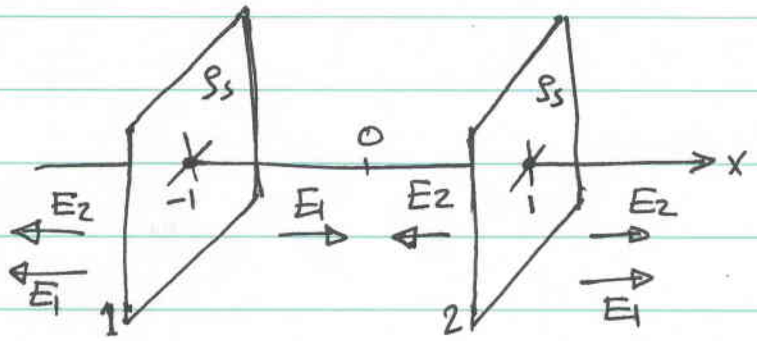
$$d\bar{F} = 9\times10^{9} \times 50\times10^{-6}\,(\underbrace{\rho_s}_{2\times10^{-5}}\,\rho\,d\rho\,d\phi) \cdot \frac{(5\bar{a}_z)}{(\rho^2+25)^{3/2}}$$

$$\therefore\ \bar{F} = 45\int_0^5 \frac{\rho\,d\rho}{(\rho^2+25)^{3/2}}\int_0^{2\pi} d\phi$$

$$= 90\pi\left[\frac{-1}{\sqrt{(\rho^2+25)}}\right]_0^5 \bar{a}_z$$

$$\therefore\ \bar{F} = 16.56\ \bar{a}_z\ N.$$

Example: Two infinite uniform sheets of charge, each with density ρ_s, are located at $x = \pm 1$. Determine $\bar{E}$ in all regions.



Solution:

for region $x > 1$

$$\bar{E}_1 = \frac{\rho_s}{2\epsilon_0}\bar{a}_x \quad , \quad \bar{E}_2 = \frac{\rho_s}{2\epsilon_0}\bar{a}_x$$

$$\therefore \bar{E}_t = \bar{E}_1 + \bar{E}_2 = \frac{\rho_s}{\epsilon_0}\bar{a}_x \ (V/m)$$

for region $x < -1$

$$E_1 = \frac{\rho_s}{2\epsilon_0}(\bar{a}_x) \quad , \quad E_2 = \frac{\rho_s}{2\epsilon_0}(-\bar{a}_x)$$

$$\therefore \bar{E}_t = \bar{E}_1 + \bar{E}_2 = \frac{\rho_s}{\epsilon_0}(-a_x). \ (V/m)$$

for region $-1 < x < 1$

$$E_t = E_1 + E_2 \Rightarrow E_t = \frac{\rho_s}{2\epsilon_0}(-a_x) + \frac{\rho_s}{2\epsilon_0}(a_x) = 0$$

$$\therefore \bar{E}_t = \begin{cases} -\dfrac{\rho_s}{\epsilon_0} a_x & x < -1 \\ 0 & -1 < x < 1 \\ \dfrac{\rho_s}{\epsilon_0} a_x & x > 1 \end{cases}$$

HW.// Determine $\bar{E}$ at $(2,0,2)$ m due to three standard charge distribution as follows:- a uniform sheet at $x=0$ m with $\rho_{s_1} = \left(\frac{1}{3\pi}\right)$ nC/m^2

a uniform sheet at $x=4$ m with $\rho_{s_2} = (-1/3\pi)$ nC/m^2 and uniform Line at $x=6$ m, $y=0$ m with $\rho_L = -2$ nC/m.

Ans: $\bar{E} = 21\,\bar{a}_x$ V/m.